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TASI LECTURES ON COMPACTIFICATION AND DUALITY

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We describe the moduli spaces of theories with 32 or 16 supercharges, from several points of view. Included is a review of backgrounds with D-branes (including type I' vacua and F-theory), a discussion of holonomy of Riemannian metrics, and an introduction to the relevant portions of algebraic geometry. The case of K3 surfaces is treated in some detail.

Introduction

Although superstring theories themselves are quite restricted in number, and naturally formulated in ten (spacetime) dimensions, there is a wide range of possible effective theories in lower dimension which are obtained by compactifying these theories. One of the remarkable features of this story is that such effective theories can often be realized in more than one way as compactified string theories, a phenomenon referred to as *duality*. Physical parameters such as the string coupling are different in the dual descriptions. Thus, in the parameter space, or “moduli space,” for the set of theories of a given type, there will be regions where one or another of the dual descriptions can be studied more easily. For example, the effective string coupling may become weak, leading to the possibility of studying the theory perturbatively in an appropriate limit (or boundary point) of the moduli space.

In these lectures, we focus on compactifications which have 16 or 32 supercharges (a property shared by the original ten-dimensional theories). Compactifications with less supersymmetry are treated in Paul Aspinwall's lectures in this volume.

We begin with a general discussion of dualities in lecture I, working with flat spacetimes only, in which some dimensions have been compactified into a torus. One of the surprising non-perturbative features is the emergence of M-theory, but there are other interesting dualities as well.

Not all limiting directions in the moduli spaces can be studied in this way, so in lecture II we are led to introduce D-branes into our superstring backgrounds, applying T-duality to type I string theory. A detailed analysis of the corresponding theories leads to type I' theory, to F-theory, and to M-theory compactified on a special type of curved manifold, the K3 surfaces.

In lecture III we introduce the general problem of using curved manifolds as superstring backgrounds, discuss the holonomy classification of Riemannian manifolds, and are quickly led to introduce the tools of algebraic geometry for the study of these manifolds. We give a detailed review of the relevant portion of algebraic geometry.

Finally, in lecture IV we return to the case of K3 surfaces and complete the story of compactifications with 16 supercharges.

The reader should be familiar with perturbative string theory, as presented for example in the textbooks of Green, Schwarz, and Witten¹ or of Polchinski.² Good general references for lectures I and II are Polchinski's text and a review by Sen.³ (We also refer the reader to a review by Mukhi,⁴ and a more comprehensive review by Sen⁵ for additional details.) A good general reference for lecture III is the book of Griffiths and Harris;⁶ in addition, much of the algebraic geometry relevant to string compactification is discussed in the book of Cox and Katz.⁷ Further details about K3 surfaces, as discussed in lecture IV, are available from either a mathematical⁸ or physical⁹ perspective.

Lecture I: S, T, U and all that

1 Perturbative superstring theories

There are five superstring theories. Each is naturally formulated in ten dimensions, and can be studied perturbatively at weak string coupling by means of conformal field theory.

The five cases are:

Type I. A theory of open and closed strings, coupled to gauge fields taking values in $\mathfrak{so}(32)$. (The global gauge group of the perturbative theory is¹⁰ $O(32)/\{\pm 1\}$.) This theory has 16 supercharges.

Types IIA and IIB. A theory of closed strings only (in the perturbative description), with abelian gauge symmetry in type IIA and no gauge symmetry in type IIB. The bosonic spectrum includes “Neveu–Schwarz–Neveu–Schwarz,” or NS-NS, fields (a graviton, a scalar field called the dilaton, and a two-form field), as well as additional “Ramond–Ramond,” or R-R, p -form fields, where p is odd for type IIA and even for type IIB. These theories have 32 supercharges.

Types HE and HO. A theory of (heterotic) closed strings only, coupled to gauge fields taking values in $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ in type HE, and $\mathfrak{so}(32)$ in type HO. The global gauge groups are $(E_8 \times E_8) \ltimes \mathbb{Z}_2$ for type HE, and ^a $\text{Spin}(32)/\mathbb{Z}_2$ for type HO. The type HE and HO theories each have 16 supercharges.

2 S duality and strong coupling limits

The duality revolution in string theory began with the realization that strong-coupling limits of the five superstring theories could be analyzed if certain non-perturbative effects were taken into account. These effects are the result of *D-branes*, which are massive

^aThe notation $\text{Spin}(32)/\mathbb{Z}_2$ denotes a quotient of $\text{Spin}(32)$ by a nontrivial \mathbb{Z}_2 in the center which does *not* yield $\text{SO}(32)$. Since the center of $\text{Spin}(32)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$, there are two such quotients, but they are isomorphic.

BPS states in the type I and II theories that couple to R-R fields. A Dp -brane is an object in spacetime with p spatial and 1 time dimension, on which open strings can end. Some D-branes become light at strong coupling, where they provide the fundamental degrees of freedom for a dual formulation of the theory.

We consider the strong coupling behavior case by case.

Type IIB. The type IIB theory has two scalar fields: the dilaton, and the R-R zero-form. The dilaton couples to the fundamental string of the theory, while the R-R zero-form couples to the D-string (another name for the D1-brane). At strong coupling, this D-string becomes light—the lightest thing in the spectrum—and exhibits the characteristics of a type IIB string. The conclusion is that the type IIB theory has a weak-strong duality, called *S-duality*.

This conclusion is further bolstered by consideration of the type IIB supergravity action, which describes this theory at low energies. Letting Φ denote the dilaton, $G_{\mu\nu}$ the graviton measured in “string frame,” B_2 the NS-NS two-form field, and C_p the R-R p -form field, the action is invariant under the S-duality transformation

$$\Phi \mapsto -\Phi, \quad G_{\mu\nu} \mapsto e^{-\Phi} G_{\mu\nu}, \quad B_2 \mapsto C_2, \quad C_2 \mapsto -B_2, \quad C_4 \mapsto C_4 \quad (1)$$

(setting $C_0 = 0$ for simplicity).

This symmetry looks more natural if written in “Einstein frame” rather than string frame: the Einstein frame graviton is $e^{-\Phi/2} G_{\mu\nu}$, which is *invariant* under the S-duality transformation given by Eq. (1).

In string theory, the Ramond-Ramond fields are invariant under periodic shifts; in particular, the shift $C_0 \rightarrow C_0 + 1$ leaves the theory invariant. This combines with the S-duality to give an $SL(2, \mathbb{Z})$ symmetry of the type IIB theory.¹¹

Note that the supergravity action is invariant under an action of $SL(2, \mathbb{R})$. But the R-R shifts can only be by integral amounts in string theory, so we expect precisely $SL(2, \mathbb{Z})$ as the symmetry of the type IIB string theory.

Type I. There is again a D1-brane in this theory which becomes light at strong coupling. However, in this case we see the behavior of a heterotic string in the strong coupling limit, rather than type I. So the weak-strong duality relates two *different* theories: type I and type HO.^{12,13}

It turns out that non-perturbative effects in type I alter the gauge group¹⁰ from $O(32)/\{\pm 1\}$ to $Spin(32)/\mathbb{Z}_2$, which thus agrees with the (perturbative) gauge group of type HO theory. This is explained in John Schwarz’s lectures in this volume.

Type IIA. We get a different behavior this time, due to the light objects being D-particles, i.e., D0-branes. It is believed that there exist bound states of n D0-branes for every n . Such a bound state will have mass $n/g\sqrt{\alpha'}$. As $g \rightarrow \infty$, this tower of states approaches a continuous spectrum whose natural explanation comes from Kaluza–Klein reduction on a *extra* circle of radius $g\sqrt{\alpha'}$. Thus we are led to the conclusion that the strong coupling limit of type IIA string theory is a mysterious eleven-dimensional theory, known as “M-theory”.^{14,12} It is not a string theory, but it does have a low-energy description in terms of eleven-dimensional supergravity. The bosonic field content of M-theory is quite simple, consisting of a graviton and a three-form field.

Types HE and HO. We cannot directly analyze the strong coupling limits in these cases with D-brane technology. However, we can infer from the above discussion that the strong coupling limit of the type HO string is the type I string. We will discuss the strong coupling limit of the type HE string theory in the next lecture.

3 T-duality for type II theories

Another important duality relating string theories is known as *T-duality*. T-duality has a non-trivial effect on the perturbative string, and has been known for much longer than the S-dualities described in the previous section.

T-duality appears when the spacetime on which the string theory is being formulated includes a compact circle S^1 . A string wrapped on a circle (or more generally, on a torus $T^d = (S^1)^d$) has *winding modes* as well as the conventional momentum modes. In the perturbative analysis, by using a Fourier transform, it can be seen that the conformal field theory is invariant under

$$r \rightarrow \alpha'/r, \quad \text{momentum} \rightarrow \text{winding}, \quad \text{winding} \rightarrow \text{momentum}. \quad (2)$$

(Here, r is the radius of the circle and α' is the string tension.) This remarkable result relating large and small distances was regarded as the first concrete evidence that string theory must modify our traditional understanding of geometry.

In this section, we discuss T-duality for the type II theories; we shall return to T-duality in the case of heterotic theories in section 5, and in the case of type I theory in lecture II.

The worldsheet action for strings on a torus depends on a choice of flat metric on the torus, and a choice of NS-NS two-form field (the “B-field”). We can separate out the volume as a separate parameter, and recall that the space of volume-one flat metrics on a torus can be described as $\text{SL}(d)/\text{SO}(d)$. The entire parameter space is thus

$$\Gamma_0 \backslash \Lambda^2 \mathbb{R}^d \times \mathbb{R}^+ \times \text{SL}(d)/\text{SO}(d) \quad (3)$$

with discrete identifications Γ_0 coming from two sources: diffeomorphisms of T^d (which contribute $\text{SL}(d, \mathbb{Z})$) and integral shifts of the B-field (which contribute $\Lambda^2 \mathbb{Z}^d$). The total discrete group coming from this geometrical analysis is $\Gamma_0 = \Lambda^2 \mathbb{Z}^d \ltimes \text{SL}(d, \mathbb{Z})$.

When T-duality is included, this group becomes much larger: in fact, it enlarges to $\text{O}(\Lambda^{d,d})$, where $\Lambda^{d,d}$ denotes a lattice with inner product of signature (d, d) , which is *even* and *unimodular*. (We are employing standard mathematical terminology here: a “lattice” has a bilinear pairing $\langle \ell_1, \ell_2 \rangle$ taking values in \mathbb{Z} , “even”

means that $\langle \ell, \ell \rangle$ is in $2\mathbb{Z}$ for every $\ell \in \Lambda$, and “unimodular” means that for every $\ell_1 \in \Lambda$, there is some $\ell_2 \in \Lambda$ such that $\langle \ell_1, \ell_2 \rangle = 1$.) It is known¹⁵ that $\Lambda^{d,d}$ must be isomorphic to the lattice whose bilinear pairing has matrix

$$\begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix} \quad (4)$$

in an appropriate basis, where I_d is the $d \times d$ identity matrix.

The most elegant formulation of all of this, essentially due to Narain,¹⁶ exploits the isomorphism

$$\Lambda^2 \mathbb{R}^d \times \mathbb{R}^+ \times \mathrm{SL}(d)/\mathrm{SO}(d) \cong \mathrm{O}(d,d)/(\mathrm{O}(d) \times \mathrm{O}(d)) \quad (5)$$

to write the moduli space in the form

$$\mathrm{O}(\Lambda^{d,d}) \backslash \mathrm{O}(d,d)/\mathrm{O}(d) \times \mathrm{O}(d). \quad (6)$$

Now we wish to extend this analysis to string theory, going beyond perturbation theory. The first remark concerns the Ramond-Ramond fields: the scalars in our effective theory which come from the R-R sector essentially transform in one of the spinor representations of $\mathfrak{o}(d,d)$, in type II theories. (More precisely, the R-R scalars must be modified by the addition of some NS-NS and mixed states before they transform in this way.¹⁷) Moreover, the vectors in a type II theory transform in the *other* spinor representation. Thus, we learn that the appropriate symmetry group for the moduli space must be $\mathrm{Spin}(d,d)$ (rather than $\mathrm{SO}(d,d)$ or some intermediate group), since *both* spinor representations must be representations of this group. This makes a small change in the description of the moduli space, which should be described as

$$\mathrm{Spin}(\Lambda^{d,d}) \backslash \mathrm{Spin}(d,d)/\mathrm{Spin}(d) \times \mathrm{Spin}(d), \quad (7)$$

but that actually agrees with the previous description in Eq. (6).

Moreover, when comparing type IIA and IIB theories, we find that the spinor representations associated to the R-R scalars and to the vectors are *reversed* by T-duality; that is, the T-duality map interchanges types IIA and IIB.

There is a potential difficulty in the above analysis when the rest of the spacetime is not flat, as was recently stressed by Aspinwall and Plesser.¹⁸ One way to think about this difficulty is to notice that we have a relatively small group $SL(d, \mathbb{Z}) \ltimes \Lambda^2 \mathbb{Z}^d$ and a small number of T-duality transformations which together generate a specific larger group $O(\Lambda^{d,d})$. In order for this to work, many group-theoretical identities involving $SL(d, \mathbb{Z}) \ltimes \Lambda^2 \mathbb{Z}^d$ and T-dualities must hold. But if the moduli spaces in question have less supersymmetry and become subject to instanton corrections, these identities may fail to hold and the generated group will be much larger. This is reminiscent of a familiar phenomenon when studying symmetries of a quantum field theory: it may be that the symmetry algebra can be defined by symmetries which extend off-shell, but the *relations* in the symmetry algebra only hold on-shell. (One often says in this situation that the algebra “closes on-shell.”) The conclusion is that T-duality holds in the expected form when there is a large amount of supersymmetry, but in vacua where some of the supersymmetry is broken, T-duality may also break down.

4 U-duality

If we put together what we have learned about S-duality and T-duality for type II theories in nine dimensions, we arrive at the following picture: starting with M-theory, compactify on T^2 with r_9, r_{10} being the radii of the circles, and consider limits when r_j gets large or small (illustrated in Figure 1).

There is a symmetry $r_9 \leftrightarrow r_{10}$ which is geometric on the M-theory side, which generates $SL(2, \mathbb{Z})$ on the type IIB side, and which shows that you get the *same* type IIA theory no matter

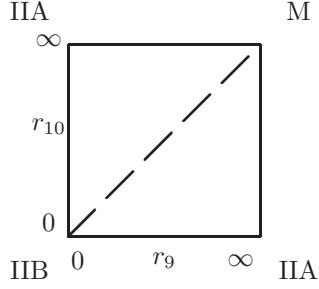


Figure 1: Compactifications of M-theory

which M-theory circle you shrink down.^{19,20}

One feature of this picture which will be important later: the parameters of the type IIA theory can be written as

$$g_{IIA} = r_{10}^{3/2}, \quad r_{IIA} = r_{10}^{1/2} r_9 \quad (8)$$

(measuring the radius in string frame), and the parameters of the type IIB theory in string frame become:

$$\begin{aligned} g_{IIB} &= r_{IIA}^{-1} g_{IIA} = (r_{10}/r_9)^{1/2} \\ r_{IIB} &= 1/r_{IIA} = r_{10}^{-1/2} r_9^{-1} \end{aligned} \quad (9)$$

and so in Einstein frame, we get

$$r_{IIB, \text{Einstein}} = g_{IIB}^{-1/2} r_{IIB} = (r_9 r_{10})^{-3/4}. \quad (10)$$

More generally, we can study the type IIA and IIB theories compactified on T^d by means of M-theory compactified on T^{d+1} . The massless bosonic fields in the effective theory are derived from the graviton and three-form field in eleven dimensions. One thing which must be specified is a flat metric on T^{d+1} (the expectation

value of the graviton in the compact directions). These are parameterized by

$$\text{Met}(T^{d+1}) = \Gamma_0 \backslash \mathbb{R}^+ \times \text{SL}(d+1, \mathbb{R}) / \text{SO}(d+1), \quad (11)$$

where $\Gamma_0 = \text{SL}(d+1, \mathbb{Z})$ comes from diffeomorphisms of T^{d+1} . The rest of the fields transform in representations of $\text{SL}(d+1, \mathbb{R})$, and we will label them accordingly.

The massless bosonic field content arises from two sources. As noted above, the M-theory graviton contributes scalars parameterized by $\text{Met}(T^{d+1})$, together with $d+1$ vectors and a lower-dimensional graviton. On the other hand, the M-theory three-form contributes $\binom{d+1}{3}$ scalars, $\binom{d+1}{2}$ vectors, $d+1$ two-form fields, and a single three-form field. The group of discrete identifications is enlarged from $\Gamma_0 = \text{SL}(d+1, \mathbb{Z})$ to include a periodicity of $\Lambda^3 \mathbb{Z}^{d+1}$ on the scalars coming from the M-theory three-form. (As we shall see below, a further enlargement is in fact expected.)

From the type IIA perspective, the symmetry group $\text{SL}(d, \mathbb{R})$ is enhanced to $\text{O}(d, d)$ through T -duality, or to $\text{SL}(d+1, \mathbb{R})$ through the M-theory interpretation. Together, these symmetries generate the larger *U-duality group*. To see what it is, consider the Dynkin diagram of $\text{SL}(d, \mathbb{R})$, which is A_{d-1} (with $d-1$ nodes):



The enlargement to $\text{O}(d, d)$ has Dynkin diagram D_d :



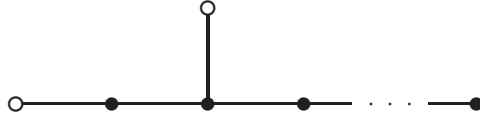
and the enlargement to $\text{SL}(d+1, \mathbb{R})$ is A_d (with d nodes):



Table 1: Field content of compactified M-theory

$d+1$	flat metrics on T^{d+1}	additional scalars	vectors	two- forms
0	$\{1\}$			
1_A	\mathbb{R}^+		$\underline{1}$	$\underline{1}$
1_B	(Type IIB)	$\underline{2}$		$\underline{2}$
2	$\mathbb{R}^+ \times \text{SL}(2, \mathbb{R}) / \text{SO}(2)$		$\underline{2} \oplus \underline{1}$	$\underline{2}$
3	$\mathbb{R}^+ \times \text{SL}(3, \mathbb{R}) / \text{SO}(3)$	$\underline{1}$	$\underline{3} \oplus \underline{3}$	$\underline{3}$
4	$\mathbb{R}^+ \times \text{SL}(4, \mathbb{R}) / \text{SO}(4)$	$\underline{4}$	$\underline{4} \oplus \underline{6}$	$\underline{4} \oplus \underline{1}$
5	$\mathbb{R}^+ \times \text{SL}(5, \mathbb{R}) / \text{SO}(5)$	$\underline{10}$	$\underline{5} \oplus \underline{10} \oplus \underline{1}$	$\underline{5}$
6	$\mathbb{R}^+ \times \text{SL}(6, \mathbb{R}) / \text{SO}(6)$	$\underline{20} \oplus \underline{1}$	$\underline{6} \oplus \underline{15} \oplus \underline{6}$	
7	$\mathbb{R}^+ \times \text{SL}(7, \mathbb{R}) / \text{SO}(7)$	$\underline{35} \oplus \underline{7}$	$\underline{7} \oplus \underline{21}$	
8	$\mathbb{R}^+ \times \text{SL}(8, \mathbb{R}) / \text{SO}(8)$	$\underline{56} \oplus (\underline{8} \oplus \underline{28})$		

leading to a combined diagram E_{d+1} (with $d+1$ nodes):



(Actually, there is another possible combination of D_d and A_d , yielding D_{d+1} , if the opposite end of D_d is lengthened, but the other fields we have which transform under D_d and A_d do so in a way which rules out that combination.)

The interpretation of E_{d+1} for small values of d is subtle, and we have collected all of the necessary data into two Tables. In Table 1, we show the various contributions to the scalar and vector field content of each of the theories,^{21,22} and in Table 2 we indicate how these are assembled into a symmetric space G/K . Each Table includes an entry for the type IIB theory in ten dimensions as well as the various compactifications of M-theory.

The “additional” scalar fields come from two sources. First, as

noted earlier, the M-theory three-form contributes $\binom{d+1}{3}$ scalars. In addition, when the effective dimension $11 - (d + 1)$ is small, other scalars can arise as duals of p -form fields. So when $d + 1 = 6$, the three-form dualizes to a scalar, and when $d + 1 = 7$, the seven two-forms dualize to scalars. (We shall discuss the case $d + 1 = 8$ momentarily.)

Similarly, the vectors in the effective theory come from three sources: Kaluza–Klein vectors, vectors from the three-form, and “dual” vectors (coming from dualizing other fields). So when $d + 1 = 5$, the three-form dualizes to a vector, and when $d + 1 = 6$, the six two-forms dualize to vectors.

In the case $d + 1 = 7$, there are no additional vectors which arise in this way. When we go to the case $d + 1 = 8$, *all* of the vectors can be dualized to scalars; we do this, and treat both the 8 Kaluza–Klein vectors and the 28 vectors from the three-form as “additional” scalars. All of this is indicated in Table 1.

The parameter spaces take the form $\Gamma \backslash G/K$ where $G = E_{d+1(d+1)}$ is a noncompact group, K is a maximal compact subgroup, and Γ is a discrete group. (We have listed the simply-connected spaces G/K in Table 2.) Remarkably, the scalars and vectors assemble themselves into representations of $G = E_{d+1(d+1)}$ in every case.

The non-compact groups which are appearing here are so-called “split forms.” In general, complex semisimple Lie algebras have a classification by Dynkin diagrams, and there is a unique connected, simply connected complex group for each algebra. (These are groups like $\mathrm{SL}(n, \mathbb{C})$ or the universal cover of $\mathrm{SO}(n, \mathbb{C})$.) There are a number of different real groups whose complexification is a given complex group. The most familiar ones to physicists are the compact groups (such as $\mathrm{SU}(n)$ and $\mathrm{SO}(n)$ in the examples above.) But there are also a number of non-compact groups with the same complexification, such as $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SO}(p, q)$. The “split forms” are the real groups which are as far from compact as possible.

The discrete groups Γ are believed to be integer versions of these

Table 2: The moduli space of compactified M-theory

$d + 1$	G/K
0	$\{1\}$
1_A	\mathbb{R}^+
1_B	$\text{SL}(2, \mathbb{R})/\text{SO}(2)$
2	$\mathbb{R}^+ \times \text{SL}(2, \mathbb{R})/\text{SO}(2)$
3	$\text{SL}(2, \mathbb{R}) \times \text{SL}(3, \mathbb{R})/\text{SO}(2) \times \text{SO}(3)$
4	$\text{SL}(5, \mathbb{R})/\text{SO}(5)$
5	$\text{SO}(5, 5)/\text{SO}(5) \times \text{SO}(5)$
6	$E_{6(6)}/\text{Sp}(4)$
7	$E_{7(7)}/\text{SU}(8)$
8	$E_{8(8)}/\text{SO}(16)$

split groups.¹¹ The full group G is a symmetry of the compactified supergravity,²³ but string or M-theory should break this to Γ . It is believed that in each dimension other than ten, the parameter space for theories with 32 supercharges is connected, and is precisely the space $\Gamma \backslash G/K$ described above. (In dimension ten, the type IIA and type IIB theories provide different connected components, which we have labeled as 1_A and 1_B in Table 2.)

5 Heterotic T-duality

Returning again to T-duality, we wish to discuss T-duality for heterotic strings.

The heterotic string theories include gauge fields in the NS-NS sector, and the Narain analysis is modified by their presence. When compactifying on T^d , Wilson lines for these gauge fields are among the parameters. We will only consider vacua for which the gauge bundle has trivial topology, and with the property that the Wilson lines can be simultaneously conjugated into a Cartan torus. (This latter property always holds when $d \leq 2$.) Imposing these proper-

ties leads us to an irreducible component of the moduli space, called the *standard component*, in each dimension less than ten. There are many ways to construct other components; we will discuss these briefly at the end of this section.

In ten dimensions, there are two kinds of heterotic theory, and we can represent the Cartan torus of the gauge group from the ten-dimensional theory in the form $(L_G \otimes \mathbb{R})/L_G$, where L_G is the root lattice of the gauge group.^b The two possible root lattices will be denoted $L_{E_8} \oplus L_{E_8}$ and $L_{(\text{Spin } 32)/\mathbb{Z}_2}$; each is an even, unimodular lattice of rank 16.

Our previous parameter space $O(d, d)/O(d) \times O(d)$ must be supplemented by $L_G \otimes \mathbb{R}^d$ to include the Wilson lines, with an initial duality group of

$$(\text{SL}(d, \mathbb{Z}) \rtimes \Lambda^2 \mathbb{Z}^d) \rtimes L_G \otimes \mathbb{Z}^d \quad (12)$$

(suppressing the string coupling).

Again, there is an elegant version of the parameter space essentially due to Narain:^{16, 24}

$$O(d, d)/O(d) \times O(d) \times L_G \otimes \mathbb{R}^d \cong O(d, d+16)/O(d) \times O(d+16) \quad (13)$$

and when the string coupling is included, there is an additional factor of \mathbb{R}^+ .

As indicated below, T-duality provides identifications between the two Narain moduli spaces for types HE and HO, and (when applied twice in succession) generates a larger discrete duality group $O(\Lambda^{d, d+16})$. The gauge group in the effective theory includes both momentum and winding modes around T^d , and its Cartan torus takes the form $(\Lambda^{d, d+16} \otimes \mathbb{R})/\Lambda^{d, d+16}$.

It is a useful mathematical fact¹⁵ that for indefinite lattices, there is only one even unimodular indefinite lattice for each rank

^bNote that the root lattice is insensitive to the fact that the gauge group $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ of HE theory is disconnected.

and signature (up to isomorphism). So our notation $\Lambda^{d,d+16}$ is unambiguous. Moreover, among *definite* lattices, the low rank ones can be classified: there is only one of rank 8, namely L_{E_8} , and there are exactly two of rank 16, namely $L_{E_8} \oplus L_{E_8}$ and $L_{\text{Spin}(32)/\mathbb{Z}_2}$. The theorem guarantees that

$$L_{E_8} \oplus L_{E_8} \oplus \Lambda^{d,d} \cong \Lambda^{d,d+16} \cong L_{\text{Spin}(32)/\mathbb{Z}_2} \oplus \Lambda^{d,d} \quad (14)$$

whenever $d > 0$.

Let us consider T-duality in the case $d = 1$. The space

$$\text{O}(\Lambda^{1,17}) \backslash \text{O}(1, 17) / \text{O}(1) \times \text{O}(17) \quad (15)$$

has exactly two asymptotic boundary points, one associated to the decomposition $\Lambda^{1,17} \cong L_{E_8} \oplus L_{E_8} \oplus \Lambda^{1,1}$, and the other to the decomposition $\Lambda^{1,17} \cong L_{\text{Spin}(32)/\mathbb{Z}_2} \oplus \Lambda^{1,1}$. (The string coupling is small in both cases, and we are suppressing it.) We assign the boundary points the interpretations of types HE and HO strings, or large radius and small radius. T-duality will relate these interpretations. (See Polchinski,² vol. 2, p. 78 for details.)

In fact, starting from either heterotic theory, there is a simple choice of Wilson line (a group element of order two, in fact) which breaks the gauge algebra to²⁵ $\mathfrak{so}(16)^{\oplus 2} \oplus \mathfrak{u}(1)^{\oplus 2}$. Globally, the gauge group becomes²⁶ $(\text{Spin}(16)^2 \times U(1)^2)/\mathbb{Z}_2$ for either theory. If we leave that group unbroken, then the only remaining parameter is the radius. An analysis of the massive states shows that if we map $r \rightarrow 1/r$ while exchanging momentum and winding modes, then the two heterotic theories are exchanged.

This leads to a picture in nine dimensions similar to the one found for the case of 32 supercharges, and illustrated in Figure 2.

We will investigate the missing corner in the next lecture. For later use, we record the relationships among couplings:

$$\begin{aligned} g_I &= g_{HO}^{-1}, & r_{9,I} &= g_{HO}^{-1/2} r_{9,HO}; \\ g_{HE} &= g_{HO} r_{9,HO}^{-1}, & r_{9,HE} &= r_{9,HO}^{-1}. \end{aligned} \quad (16)$$

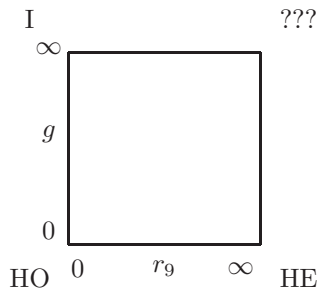


Figure 2: Compactifications of Type I and Heterotic Strings

The first line comes from S-duality, and the second line from T-duality.

The string coupling is in fact the only additional parameter in the type HE and HO theories not present in the perturbative analysis, when $d \leq 4$. In that case, the full moduli space is

$$\mathrm{O}(\Lambda^{d,d+16}) \backslash \mathbb{R}^+ \times \mathrm{O}(d, d+16) / \mathrm{O}(d) \times \mathrm{O}(d+16) \quad (17)$$

(including the string coupling), and the vectors in the theory transform in the vector representation of $\mathrm{O}(d, d+16)$. In lower effective, we get further fields in the non-perturbative analysis, as in the M-theory case: when $d+1 = 6$ the two-form dualizes to give an extra vector, when $d+1 = 7$ the two-form dualizes to give an extra scalar, and when $d+1 = 8$ all of the vectors can be dualized to scalars. As in the M-theory case, the fields assemble themselves into highly symmetric spaces, as indicated in Table 3.

In addition to the standard component, there are many other components of the moduli space of theories with 16 supercharges. For example, there is a construction known as the CHL string²⁷ which exists in dimension less than ten. In dimension nine, the CHL string can be described as the HE string compactified on a

Table 3: The standard component of the moduli space of compactified heterotic string theory

$d + 1$	standard component
1_E	\mathbb{R}^+
1_O	\mathbb{R}^+
2	$O(\Lambda^{1,17}) \backslash \mathbb{R}^+ \times O(1, 17) / O(1) \times O(17)$
3	$O(\Lambda^{2,18}) \backslash \mathbb{R}^+ \times O(2, 18) / O(2) \times O(18)$
4	$O(\Lambda^{3,19}) \backslash \mathbb{R}^+ \times O(3, 19) / O(3) \times O(19)$
5	$O(\Lambda^{4,20}) \backslash \mathbb{R}^+ \times O(4, 20) / O(4) \times O(20)$
6	$O(\Lambda^{5,21}) \backslash \mathbb{R}^+ \times O(5, 21) / O(5) \times O(21)$
7	$(SL(2, \mathbb{Z}) \times O(\Lambda^{6,22})) \backslash \mathfrak{h} \times O(6, 22) / O(6) \times O(22)$
8	$O(\Lambda^{8,24}) \backslash O(8, 24) / O(8) \times O(24)$

circle with a Wilson line implementing the \mathbb{Z}_2 gauge symmetry which exchanges the E_8 factors.^{28,29} (This gives a new component in nine dimensions, since that gauge transformation cannot be conjugated into a Cartan torus.) In eight dimensions and below, the CHL component can alternatively be described as the HO string compactified on a circle with a non-trivial gauge bundle, the bundle “without vector structure.”³⁰ (These two descriptions are related by T-duality, as in the case of the standard component.) The CHL component has been studied from many points of view,^{31,30,32–34} in dimension nine, its moduli space takes the form

$$O(\Lambda^{1,9}) \backslash \mathbb{R}^+ \times O(1, 9) / O(1) \times O(9). \quad (18)$$

There are numerous other components in lower dimension, which can be constructed with a variety of different techniques.^{27,20,35,30,36}

Lecture II: Backgrounds with Branes

6 Type I theory as an orientifold

The type I string theory can be described as an “orientifold” of the type IIB theory. This means that one introduces the orientation-reversal operator Ω which reverses the orientation of the world-sheet, and projects to the set of invariant states, similar to an orbifold projection. The analogue of the twisted sector in orbifold theory is provided by new degrees of freedom which can be described as an *orientifold O9-plane* together with 16 *D9-branes* (projected from 32 D9-branes in type IIB theory). A collection of 32 space-filling D9-branes in type IIB theory would have $SU(32)$ gauge symmetry via the Chan–Paton mechanism, but the Chan–Paton factors are restricted by the orientifold projection to take values in $SO(32)$.

In this lecture, we will study models obtained by compactifying type I on a torus T^k and performing T-duality on T^k (dualizing all compact directions simultaneously). As one application of this study, we will find a weakly coupled dual description of the strong coupling limit of the type HE string; another application will be to so-called F-theory models. For these applications, we begin with the type HO string compactified on T^k , apply S-duality to get to the type I string, then apply T-duality and determine a weakly coupled description of the corresponding theory.

Since the type I theory contains open strings, the T-dual theory will acquire branes at which the open strings may end—these are just the standard T-duals of the original D9-branes, and give Dp -branes in the dual theory (where $p = 9 - k$). In addition, the T-dual of the orientifold operator Ω is Ω times a reflection which reverses the T-dualized coordinates, i.e., $\iota : (x_1, \dots, x_k) \rightarrow (-x_1, \dots, -x_k)$. There are “orientifold Op -planes” located at the fixed points of T^k/ι . (We label an orientifold plane according to the number of spatial dimensions it occupies, just like with a D-brane. Thus, a

Dp -brane and an Op -plane each occupy p spatial dimensions, i.e., $p + 1$ spacetime dimensions.)

The locations of the Dp -branes in the dual theory are encoded by the Wilson lines around T^k in the original theory. So for general Wilson line values, the dual theory has 2^k Op -planes and 16 Dp -branes deployed at various locations in T^k/ι . The Op -planes have Dp -brane charge -2^{4-k} each, so the total Dp -brane charge vanishes globally, but the existence of local Dp -brane charges means that the Ramond-Ramond fields in the background will not vanish, but will vary over spacetime. Such backgrounds are difficult to describe directly.

When $k \leq 4$, we can choose special positions for the Dp -branes so that 2^{4-k} Dp -branes are located on top of each Op -plane. In this case, the Dp -brane charges cancel locally, and no Ramond-Ramond background is needed; moreover, the model can be studied at weak string coupling. The gauge algebra contains a copy of $\mathfrak{so}(2^{5-k})$ for each orientifold plane, so such brane positions must correspond to Wilson lines in the type I theory which break the nonabelian part of the perturbative gauge algebra of type I from $\mathfrak{so}(32)$ to $\mathfrak{so}(2^{5-k})^{\oplus 2^k}$.

It is possible²⁶ to analyze the non-perturbative gauge groups, for example from the heterotic perspective, to obtain the gauge groups given in Table 4 (exploiting the fact that $\text{Spin}(4) \cong \text{SU}(2)^2$). For ease of discussion, though, we shall henceforth focus on the gauge *algebras* rather than the global form of the gauge groups.

The descriptions above derive from an analysis of the conformal field theory (as discussed in Polchinski,² for example). We will, in the rest of this lecture, explore how these models are described in a supergravity approximation.

We therefore wish to study *backgrounds with branes*. We model these by using an ansatz for branes which is similar to that used in studying near-horizon limits (see e.g. the lectures by Igor Klebanov or by Amanda Peet in this volume): for a collection of Dp -branes, we consider a spacetime of the form $\mathbb{R}^{p,1} \times Y^{9-p}$ with a metric of

Table 4: Non-perturbative gauge groups

k	gauge group
0	$\text{Spin}(32)/\mathbb{Z}_2$
1	$(\text{Spin}(16)^2 \times U(1)^2)/\mathbb{Z}_2$
2	$(\text{Spin}(8)^4 \times U(1)^4)/\mathbb{Z}_2^3$
3	$(\text{SU}(2)^{16} \times U(1)^6)/\mathbb{Z}_2^5$

the form

$$ds^2 = H(y)^{-1/2}(-dt^2 + dx_1^2 + \cdots dx_p^2) + H(y)^{1/2}(g_{ij}dy_i dy_j) \quad (19)$$

accompanied by a dilaton

$$e^\Phi = H(y)^{(3-p)/4} \quad (20)$$

and a Ramond-Ramond field strength

$$F = dt \wedge dx_1 \wedge \cdots \wedge dx_p \wedge d(H(y)^{-1}). \quad (21)$$

Here, $Y \subseteq \overline{Y}$ is the complement of a finite set of points $\{P_\alpha\}$ in \overline{Y} , $g_{ij}dy_i dy_j$ is an appropriate metric on Y (usually flat or Ricci-flat), and $H(y)$ satisfies

$$\Delta H(y) = \sum_{\alpha} N_{\alpha} \delta_{P_{\alpha}}, \quad (22)$$

i.e., its Laplacian is a sum of delta functions at the points P_{α} , weighted by integers N_{α} . Such a metric represents a background with N_{α} D p -branes located at P_{α} , for each α . (This ansatz must be slightly modified for D3-branes, but they will not concern us here.)

7 Orientifolds in dimension nine

We begin with the case $k = 1$, that is, we analyze the T-dual of the type I theory compactified on S^1 , choosing the Wilson line to break the gauge algebra to $\mathfrak{so}(16) \oplus \mathfrak{so}(16) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$. The dual theory is described as type IIA on S^1/\mathbb{Z}_2 with each endpoint of S^1/\mathbb{Z}_2 having an orientifold O8-plane and eight D8-branes. (We call this an *O8 + 8 D8 brane configuration*.) Since the local D8-brane charge is zero, we may use a harmonic function $H(y)$ on S^1 which is \mathbb{Z}_2 -invariant; since $H(y)$ is harmonic, it must be *linear*. That is, $H(y) = ay + b$, but then $H(y) = H(-y)$ implies $a = 0$ so $H(y)$ is constant. This leads to a conventional model with constant dilaton and no Ramond-Ramond flux; however, the space Y is a manifold with boundary, so the O8 + 8 D8 brane configuration still leaves its mark.

Now we allow the D8-branes to move away from the O8-planes (which is accomplished on the type I side by allowing the Wilson line to vary). The function H is now only *piecewise linear*, and the jumps in its slope measure the jumps in D8-brane charge from region to region in spacetime.^c

Every function of this type takes the form

$$H(y) = C - \frac{1}{2} \sum_{i=1}^{16} |y - y_i| \quad (23)$$

where $y_i \in [0, 1]$ are the locations of the branes, and $y \in [0, 1]$ is a coordinate on Y . A typical graph of such a function is shown in Figure 3. Note that the slopes at the endpoints are ± 8 , corresponding to the D8-brane charge of -8 carried by the orientifold O8-planes.

^cThe piecewise linear nature of the function can be seen directly from a spacetime analysis,¹³ or by considering either D4-brane probes³⁷ or D0-brane probes.³⁸

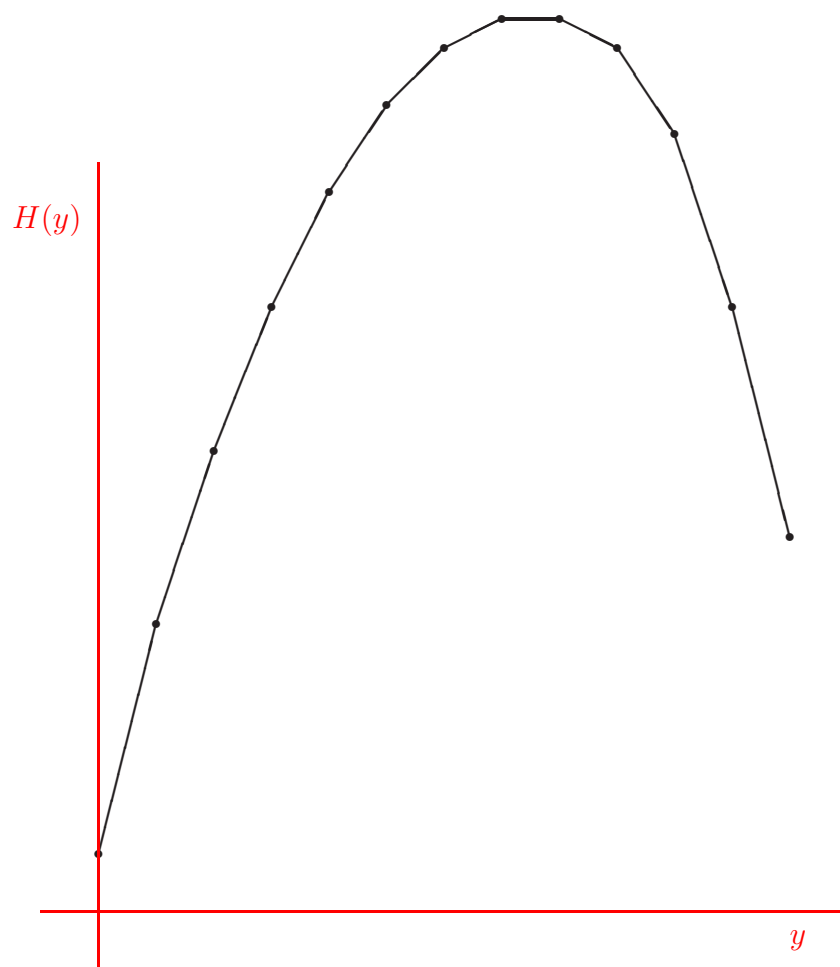


Figure 3: A generic function $H(y)$

The structure near the O8-planes can be made more transparent by extending the function $H(y)$ past those planes, i.e., defining it on S^1 in a way that is symmetric. Near $y = 0$, this can be done by rewriting

$$H(y) = \tilde{C} + 8|y| - \frac{1}{2} \sum_{i=1}^{16} (|y - y_i| + |y + y_i|). \quad (24)$$

On S^1 , we have 32 D8-branes in 16 pairs, related by reflection.

For generic locations of the D8-branes, the gauge symmetry is abelian. This is enhanced to $\mathfrak{su}(N)$ gauge symmetry when N of the D8-branes come together (i.e., their y_i values coincide), and to $\mathfrak{so}(2N)$ gauge symmetry when N of the D8-branes coincide with the O8-plane in an O8 + N D8 configuration (i.e., $y_i=0$ or $y_i=1$ for all of these). These gauge symmetry enhancements arise from open strings stretched between branes which become massless in the limit, providing off-diagonal elements of a matrix-valued gauge field.

If $N \leq 7$ D8-branes are located at $y = 0$, then the initial slope in $H(y)$ is positive and we can vary the constant C so that $H(0) = 0$, without violating the essential requirement $H(y) \geq 0$. There is a further gauge symmetry enhancement at such points³⁷ from $\mathfrak{so}(2N)$ to an algebra \mathfrak{e}_{N+1} . The new light particles^{39–41} are D0-branes bound to $y = 0$, which are light due to the locally strong coupling at that end of Y .

A more detailed exploration⁴² of the space of possible functions $H(y)$ reveals additional structure.^d There is a phase transition when strong coupling is reached (i.e., $H(0) = 0$ or $H(1) = 0$) to another set of models whose piecewise-linear functions $H(y)$ have 17 or 18 singularities. The slopes at the endpoints of $[0, 1]$ can be

^dNote that in spite of certain reservations which have been expressed about this picture,⁴³ the structure of the set of functions $H(y)$ has recently been confirmed from another point of view.⁴⁴

Table 5: Strong coupling limits of orientifold planes

$N + 1$	weak coupling theory	weak coupling gauge algebra	strong coupling theory
0	–	–	E8
1_A	$O8_A$	$\{0\}$	$(E8 + D8)_A$
1_B	$O8_B$	$\{0\}$	$(E8 + D8)_B$
2	$O8 + D8$	$\mathfrak{so}(2)$	$E8 + 2 D8$
3	$O8 + 2 D8$	$\mathfrak{so}(4)$	$E8 + 3 D8$
4	$O8 + 3 D8$	$\mathfrak{so}(6)$	$E8 + 4 D8$
5	$O8 + 4 D8$	$\mathfrak{so}(8)$	$E8 + 5 D8$
6	$O8 + 5 D8$	$\mathfrak{so}(10)$	$E8 + 6 D8$
7	$O8 + 6 D8$	$\mathfrak{so}(12)$	$E8 + 7 D8$
8	$O8 + 7 D8$	$\mathfrak{so}(14)$	$E8 + 8 D8$

as high as 9, and we interpret the local function with slope 9 as a new kind of spacetime defect: an E8-plane. In the presence of one or two E8-planes, the D8-branes are not free to take arbitrary positions, but are constrained by the requirement that $H(y) = 0$ at the E8-plane(s).

A catalog of possible behaviors is given in Tables 5 and 6. There are some irregularities in the behavior when the number of D8-branes is small: first, there are two possibilities for the pure $SU(2)$ probe theory, depending on a \mathbb{Z}_2 -valued θ -angle; we call the two kinds of spacetime defects $O8_A$ -planes and $O8_B$ -planes.^e Since $O8_A + D8$ is physically equivalent to $O8_B + D8$, this distinction is not visible for most orientifold combinations.

Second, the strong coupling limits of the $O8_A$ -plane and $O8_B$ -

^eIn a previous paper,⁴² the $O8_B$ -plane was associated with the “ D_0 theory” and the $O8_A$ -plane with the “ \widetilde{D}_0 theory,” which differ by a θ -angle.⁴⁵ Similarly, the $(E8 + D8)_B$ configuration is associated to the “ E_1 theory,” and the $(E8 + D8)_A$ configuration is associated to the “ \widetilde{E}_1 theory.”

Table 6: Gauge algebras of strong coupling limits, and del Pezzo surfaces

$N + 1$	strong coupling theory	strong coupling gauge algebra	del Pezzo surface
0	E8	$\{0\}$	\mathbb{P}^2
1_A	$(E8 + D8)_A$	$\mathfrak{u}(1)$	$\text{Bl}_1 \mathbb{P}^2$
1_B	$(E8 + D8)_B$	$\mathfrak{su}(2)$	$\mathbb{P}^1 \times \mathbb{P}^1$
2	$E8 + 2 \text{ D8}$	$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$\text{Bl}_2 \mathbb{P}^2$
3	$E8 + 3 \text{ D8}$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2)$	$\text{Bl}_3 \mathbb{P}^2$
4	$E8 + 4 \text{ D8}$	$\mathfrak{su}(5)$	$\text{Bl}_4 \mathbb{P}^2$
5	$E8 + 5 \text{ D8}$	$\mathfrak{so}(10)$	$\text{Bl}_5 \mathbb{P}^2$
6	$E8 + 6 \text{ D8}$	\mathfrak{e}_6	$\text{Bl}_6 \mathbb{P}^2$
7	$E8 + 7 \text{ D8}$	\mathfrak{e}_7	$\text{Bl}_7 \mathbb{P}^2$
8	$E8 + 8 \text{ D8}$	\mathfrak{e}_8	$\text{Bl}_8 \mathbb{P}^2$

plane give configurations we call $(E8 + D8)_A$ and $(E8 + D8)_B$. A D8-brane can be “emitted” from an $(E8 + D8)_A$ configuration to yield the exotic E8-plane itself. However, it is not possible to emit a D8-brane directly from an $(E8 + D8)_B$ configuration. As in the O8-plane case, $(E8 + D8)_A + D8$ is physically equivalent to $(E8 + D8)_B + D8$, so the distinction is again not visible for most E8-plane configurations. We only use the A or B subscript on an E8-plane in the presence of a single D8-brane.

The behavior of the function $H(y)$ is different for these two types of planes: the function for $O8 + N \text{ D8}$ has a slope of $\pm(8 - N)$ at the boundary, whereas the function for $E8 + M \text{ D8}$ has a slope of $\pm(9 - M)$.

Remarkably, the list of strong coupling gauge algebras which appear here (and which are all compact forms of appropriate “ \mathfrak{e}_{N+1} ” algebras) corresponds precisely to the list of U-duality groups in lecture I, except that here the *compact* algebras appear!

This T-dual of type I theory on S^1 is often called type I' theory

or type IA theory. Since it is related to type I by a T-duality, the couplings are related as

$$g_{I'} = g_I r_{9,I}^{-1}, \quad r_{9,I'} = r_{9,I}^{-1}. \quad (25)$$

We can now analyze the strong coupling limit of the type HE string in nine dimensions. Combining the T-duality between types HE and HO with the S-duality between type HO and type I, we get (as at the end of lecture I)

$$g_I = g_{HE}^{-1} r_{9,HE}, \quad r_{9,I} = g_{HE}^{-1/2} r_{9,HE}^{-1/2} \quad (26)$$

which is *not* a weakly coupled description for strong type HE coupling (and fixed, large $r_{9,HE}$). Since the characteristic size of $r_{9,I}$ is small, we can T-dualize to type I':

$$g_{I'} = g_{HE}^{-1/2} r_{9,HE}^{3/2}, \quad r_{9,I'} = g_{HE}^{1/2} r_{9,HE}^{1/2}. \quad (27)$$

These dualities are initially performed with gauge algebra

$$\mathfrak{so}(16)^{\oplus 2} \oplus \mathfrak{u}(1)^{\oplus 2} \quad (28)$$

which corresponds to $H(y)$ being constant; we can later tune the Wilson lines to restore $\mathfrak{e}_8 \oplus \mathfrak{e}_8 \oplus \mathfrak{u}(1)^{\oplus 2}$ on the heterotic side, and the corresponding tuning on the type I' side yields a function $H(y)$ of the form illustrated in Figure 4.

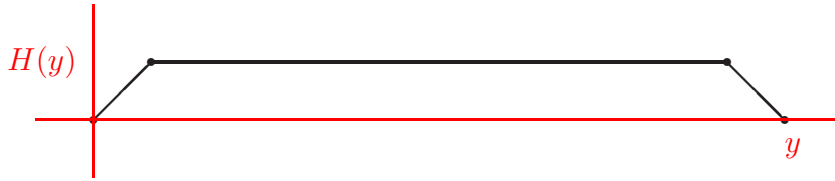


Figure 4: The function $H(y)$ for gauge algebra $\mathfrak{e}_8 \oplus \mathfrak{e}_8 \oplus \mathfrak{u}(1)^{\oplus 2}$

The type I' coupling $g_{I'} = H(y)^{-5/4}$ is strong at strong HE coupling, so we expect a description in terms of M-theory. We get M-theory scales

$$\begin{aligned} r_9 &= g_{I'}^{-1/3} r_{9,I'} = g_{HE}^{2/3} \\ r_{10} &= g_{I'}^{-2/3} = g_{HE}^{1/3} r_{9,HE}^{-1} \end{aligned} \tag{29}$$

and so we see that our strong coupling limit decompactifies the r_9 direction. This leads to the Hořava–Witten⁴⁶ picture of the strong coupling limit of the type HE string: it is described by M-theory compactified on S^1/\mathbb{Z}_2 with an E_8 gauge symmetry group bound to each end of S^1/\mathbb{Z}_2 . The exchange of the two ends leads to the additional \mathbb{Z}_2 gauge transformation of this theory.

8 Orientifolds in dimension eight, and F-theory

We turn now to the next case: the T-dual of type I on T^2 . As before, we first choose appropriate Wilson lines to break the gauge algebra to $\mathfrak{so}(8)^{\oplus 4} \oplus \mathfrak{u}(1)^{\oplus 4}$, and then perform a T-duality to obtain a type IIB string on $T^2/\mathbb{Z}_2 = S^2$, with an orientifold O7-plane and four D7-branes located at each of the four fixed points of the \mathbb{Z}_2 action. There is no local D7-brane charge near these points, but since the orientifolding operator acts as -1 on the NS-NS and R-R two-forms, there is a monodromy $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ associated with each point. Moreover, the metric on $S^2 = T^2/\mathbb{Z}_2$ is an orbifold metric, which has a “deficit angle” of π at each of the four points. (In terms of a local coordinate z , the metric takes the form $|d\sqrt{z}|^2 = \frac{1}{4}|z^{-1/2} dz|^2$ which gives deficit angle $2\pi \cdot \frac{1}{2} = \pi$.)

We now wish to vary the Wilson lines on the type I side, and study the corresponding backgrounds on the type IIB side. This problem was analyzed a number of years ago (from a slightly different perspective) by Greene, Shapere, Vafa, and Yau⁴⁷ who showed how to exploit the $\text{SL}(2, \mathbb{Z})$ symmetry to produce solutions. Since we will use $\text{SL}(2, \mathbb{Z})$, the resulting backgrounds have no conventional string theory description: they require that *different* (p, q) -

strings of the type IIB theory be fundamental at *different* points in spacetime. Nevertheless, the low energy supergravity description (with singularities along D7-branes) can be analyzed, and in fact there are other ways to view such theories as limits of string theories. Models of this general class are known as *F-theory compactifications*.^{48–51}

Note that in dimension nine we passed from a constant function to a piecewise-linear function when branes were moved away from the orientifold planes; here, we are passing from a constant function to a holomorphic function that has an $\mathrm{SL}(2, \mathbb{Z})$ -transformation property.

Since we are going to exploit the S-duality of the type IIB string in constructing these models, we should work in Einstein frame rather than string frame. The supergravity description is then in terms of a metric of the form

$$ds^2 = (-dt^2 + dx_1^2 + \cdots + dx_7^2) + H(y_1, y_2)(g_{ij}dy_i dy_j) \quad (30)$$

with dilaton

$$e^\Phi = H(y_1, y_2)^{-1} \quad (31)$$

and R-R field strength

$$F = dt \wedge dx_1 \wedge \cdots \wedge dx_p \wedge d(H(y_1, y_2)^{-1}). \quad (32)$$

The equation of motion for $H(y_1, y_2)$ is

$$\Delta H(y_1, y_2) = \sum_{\alpha} N_{\alpha} \delta_{P_{\alpha}} \quad (33)$$

as before.

Introducing a complex coordinate $z = y_1 + iy_2$, we treat the harmonic function $H(y_1, y_2)^{-1}$ as the imaginary part of a holomorphic function $\tau(z)$ (away from P_{α}):

$$H(y_1, y_2)^{-1} = \mathrm{Im} \tau(y_1 + iy_2) \quad (34)$$

where $\tau(z)$ is only well-defined up to $\mathrm{SL}(2, \mathbb{Z})$ transformations and $\mathrm{Im} \tau(z) > 0$ (since the conformal factor H is always positive).

The Greene–Shapere–Vafa–Yau solutions define H in terms of functions $\tau(z)$ which come from functions on $\mathfrak{h}/\mathrm{SL}(2, \mathbb{Z})$, where $\mathfrak{h} = \{\tau : \mathrm{Im} \tau > 0\}$ is the upper half plane, in order to get finite energy configurations. In addition to the complex field $\tau(z)$ with $\mathrm{SL}(2, \mathbb{Z})$ invariance, their solutions specify the corresponding Ricci-flat Kähler metric as

$$\frac{1}{2i} \frac{|\eta(\tau(z))|^4}{\mathrm{Im} \tau(z)} \left| \prod_{\alpha=1}^m (z - z_\alpha)^{-k_\alpha/12} \right|^2 dz d\bar{z}, \quad (35)$$

where

$$\eta(\tau) = e^{2\pi i \tau/24} \prod_n (1 - e^{2\pi i \tau n}) \quad (36)$$

is Dedekind’s eta-function.

This metric has a so-called *deficit angle*: in terms of a new variable $\tilde{z} = z^{1-k_\alpha/12}$, the metric looks conventional (and flat). But the variable \tilde{z} does not traverse a full phase as $z \mapsto e^{2\pi i} z$. The exponent $k_\alpha/12$, which determines the deficit angle of $2\pi k_\alpha/12$ at P_α , is a function of the type of singularity occurring at P_α .

In addition to the deficit angle in the metric, the function $\tau(z)$ exhibits singular behavior at each singular point: it is multi-valued near the singularity, with the multi-valuedness being given by some fractional linear transformation from $\mathrm{SL}(2, \mathbb{Z})$, which describes the change as $z \mapsto e^{2\pi i} z$.

The possible singularities in such functions $\tau(z)$ were classified by Kodaira,⁵² and are described in Table 7. Kodaira’s analysis used algebraic geometry, and we will briefly sketch it in lecture IV. The analysis directly produces the monodromy in each case, as well as providing algebro-geometric data from which the deficit angle can be determined. These are both indicated in the Table.

In addition, we have identified the gauge symmetry for each of Kodaira’s singularity types, and we have attempted to describe

Table 7: Kodaira's classification

Kodaira notation	brane configuration	gauge algebra	deficit angle	monodromy
$I_N, N > 0$	N D7	$\mathfrak{su}(N)$	$\frac{N\pi}{6}$	$\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$
I_0^*	O7 + 4 D7	$\mathfrak{so}(8)$	π	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
$I_N^*, N > 0$	O7 + (N+4)D7	$\mathfrak{so}(2N+8)$	$\pi + \frac{N\pi}{6}$	$\begin{pmatrix} -1 & N \\ 0 & -1 \end{pmatrix}$
IV^*	E7 + 6 D7	\mathfrak{e}_6	$\frac{4\pi}{3}$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$
III^*	E7 + 7 D7	\mathfrak{e}_7	$\frac{3\pi}{2}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
II^*	E7 + 8 D7	\mathfrak{e}_8	$\frac{5\pi}{3}$	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$
II	H7	$\{0\}$	$\frac{\pi}{3}$	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$
III	H7 + D7	$\mathfrak{su}(2)$	$\frac{\pi}{2}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
IV	H7 + 2 D7	$\mathfrak{su}(3)$	$\frac{2\pi}{3}$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$

each case as a “brane configuration”: the cases of N D7 branes and an $O7 + (N+4)$ D7 brane configuration follow from conventional descriptions of D7-branes and O7-planes, with enhanced gauge symmetry determined by open strings stretching between branes. There are no solutions $\tau(z)$ corresponding to an $O7 + N$ D7 brane configuration with $N < 4$.

The $E7 + N$ D7 brane configurations can be studied as strong coupling limits of $O7 + (N-1)$ D7 brane configurations, just as in nine dimensions. (In this dimension, there are no \mathbb{Z}_2 -valued θ -angles to worry about.) The $H7 + N$ D7 configurations are new to eight dimensions, and much less is known about their explicit description.

In order to get a global solution on S^2 , we need the total deficit angle to be 4π . The generic such solution has 24 D7-branes, located at distinct points. In fact, each of the $O7 + 4$ D7 brane configurations which occurred in our original orbifold splits up into six D7-branes whose positions are, however, somewhat constrained.

The function $\tau(z)$ combines the dilaton and the R-R scalar into a single holomorphic function on S^2 . To relate these F-theory compactifications to other string models, we can further compactify on S^1 , remembering that we are working in Einstein frame. As we saw in lecture I, if $r_{IIB,Einstein}$ denotes the radius of this compactification, and we compare to M-theory compactified on T^2 , we find

$$r_{IIB,Einstein} = (r_9 r_{10})^{-3/4}. \quad (37)$$

Thus, the small S^1 -limit will map over to a limit in the M-theory model in which the area of T^2 becomes large (and so the supergravity approximation should be accurate). Furthermore, we had

$$g_{IIB} = (r_{10}/r_9)^{1/2} \quad (38)$$

and more generally, τ will capture the conformal class of the metric on T^2 , which is equivalent to specifying a holomorphic structure.

There is thus a dual model in seven dimensions which takes the form of M-theory compactified on a four-manifold which is fibered by T^2 's with holomorphic structure dictated by $\tau(z)$.

The four-manifolds of this kind are not unique, but there *is* a unique one for which the holomorphic fibration has a holomorphic section. Much evidence has been amassed in favor of a proposed duality^{48,53}

$$\begin{array}{ccc} \text{M-theory on} & & \text{F-theory from} \\ \text{elliptically fibered} & \leftrightarrow & \tau(z), \text{ further} \\ \text{manifold with section} & & \text{compactified on } S^1 \\ & & \text{(with vanishing Wilson lines)} \end{array}$$

If we use instead one of the four-manifolds for which the holomorphic fibration does not have a holomorphic section, we find a model in which a Wilson line along the S^1 has been turned on.^{54,36}

9 Orientifolds in dimension seven

Finally, we consider the T-dual of type I compactified on T^3 . We again get a type IIA model on an orbifold T^3/\mathbb{Z}_2 , with an O6 + 2 D6 brane configuration located at each fixed point and a gauge algebra $\mathfrak{so}(4)^{\oplus 8} \oplus \mathfrak{u}(1)^{\oplus 6} = \mathfrak{su}(2)^{\oplus 16} \oplus \mathfrak{u}(1)^{\oplus 6}$.

If we start with the type HO string compactified on T^3 , and perform S-duality followed by T-duality, we find the following relations among couplings:

$$\begin{aligned} g_{new} &= \frac{g_{HO}^{1/2}}{r_{7,HO} r_{8,HO} r_{9,HO}} \\ r_{j,new} &= \frac{g_{HO}^{1/2}}{r_{j,HO}} \quad \text{for } j = 7, 8, 9. \end{aligned} \tag{39}$$

Thus, in the strong coupling limit of this type HO compactification, we are still seeing strong coupling in the type IIA theory, which suggests an M-theory interpretation. Lifting the orientifolding operator to M-theory reverses the sign on the tenth spatial

dimension, so the dual model is M-theory on T^4/\mathbb{Z}_2 with \mathbb{Z}_2 acting as -1 on all four coordinates. There are 16 fixed points for this action.

What becomes of the D6-branes and O6-planes in this M-theory description? Our usual ansatz near a D6-brane

$$\begin{aligned} ds^2 &= H(y)^{-1/2}(-dt^2 + dx_1^2 + \cdots + dx_6^2) + H(y)^{1/2}(g_{ij}dy_i dy_j) \\ e^\Phi &= H(y)^{-3/4} \end{aligned} \tag{40}$$

should be rewritten in M-theory frame, with metric $g^{-2/3}ds^2$:

$$d\tilde{s}^2 = (-dt^2 + dx_1^2 + \cdots + dx_6^2) + \tilde{g}_{ij}dy_i dy_j + g_{10,10}dx_{10}^2. \tag{41}$$

There is no longer a conformal factor in the metric on the world-volume of the brane, so we simply need a Ricci-flat metric on the remaining four coordinates, i.e., the D6-branes do not appear in this frame. The general solution with 16 supercharges gives a K3 surface.

In our original orbifold model, we had 16 singular points. Each can be described as a limit of a smooth K3 metric in which an S^2 has shrunk to zero area. Wrapping the M-theory membrane around such an S^2 , we see a new light particle in the limit, which lies in a vector multiplet and is responsible for $\mathfrak{su}(2)$ enhanced gauge symmetry.^f In this way, the expected $\mathfrak{su}(2)^{\oplus 16} \oplus \mathfrak{u}(1)^{\oplus 6}$ gauge algebra is reproduced.

More generally, it is possible to shrink various configurations of S^2 's to get various enhanced gauge symmetries. The spacetime singularities always take the form \mathbb{C}^2/Γ for some $\Gamma \subseteq \mathrm{SU}(2)$, and lead to the possibilities described in Table 8 (where we list the image of Γ in $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm 1\}$ due to the familiar form of the answers).

^fThe story of enhanced gauge symmetry from the point of view of D6-branes is somewhat complicated,⁵⁵ and we will not discuss it here.

Table 8: Finite subgroups of $SU(2)/\{\pm 1\}$ and the associated gauge algebras

image of Γ in $SO(3)$	classical symmetry	gauge algebra
\mathbb{Z}_N	rotations of N -gon	$\mathfrak{su}(N)$
\mathbb{D}_{2N}	rotations and reflections of N -gon	$\mathfrak{so}(2N)$
\mathbb{T}	symmetries of tetrahedron	\mathfrak{e}_6
\mathbb{O}	symmetries of octahedron or cube	\mathfrak{e}_7
\mathbb{I}	symmetries of icosahedron or dodecahedron	\mathfrak{e}_8

10 Probe-brane theories

An important verification in each of these cases is provided by the study of probe-brane theories. The branes we have studied (D8, D7, D6) are large and quite heavy. It is possible to introduce a parallel $D(p-4)$ -brane into the system as a “probe” of the background, treating the big branes as static and the small branes as fluctuating.^{56–58} The worldvolume theories on the probe-branes will have 8 supercharges (assuming that the background is otherwise flat).

In the case of D4-brane probes of D8-branes in type I', this leads to a study of five-dimensional field theories with a “Coulomb branch” of dimension one.^{37,42,45} These theories also occur on Calabi–Yau threefolds, where (for example) the contraction of a del Pezzo surface to a point yields the probe theory of an $E8 + n$ D8 brane, where $n + 1$ is the Picard number of the del Pezzo surface.

In the case of D3-brane probes of D7-branes in F-theory, many four-dimensional field theory phenomena are uncovered,⁵⁷ such as the behavior of $SU(2)$ gauge theory with $N_f < 4$ matter multiplets (from analyzing $O7 + N_f$ D7) or the existence of Argyres–Douglas type points (from analyzing $H7 + N_f$ D7). There are also exotic

branes at strong coupling with \mathfrak{e}_n gauge algebras.⁵⁹

Investigations have also been made of D2-brane probes of D6-brane theories^{60,61} Unfortunately, time does not permit us to discuss any of these matters in detail.

Lecture III: Curved Backgrounds

11 Holonomy

As we saw in the previous lecture, certain natural backgrounds for string compactification which include D-branes (and yield *singular* supergravity solutions) break half of the supersymmetry of the original theory. The other natural way to study models with reduced supersymmetry is to introduce *curved* backgrounds.

The traditional way this has been done in string theory has been to decompose the ten-dimensional spacetime as a product $X^d \times M^{1,9-d}$ of a compact manifold X and a flat spacetime M . To understand how much supersymmetry is preserved in such backgrounds, we must decompose the $(9+1)$ -dimensional spinor representation according to $\text{Spin}(d) \times \text{Spin}(1, 9-d)$, and ask how many covariantly constant spinors will exist on X^d (with respect to the given metric on X^d)—these determine the unbroken supersymmetries. (See, for example, Polchinski,² Appendix B.1.)

A variant of this construction is given by the Freund–Rubin ansatz:⁶² we make a decomposition as a product $Y^{d-1} \times \text{AdS}^{1,10-d}$ together with a nontrivial field strength for one of the supergravity fields. (Similar constructions can also be made for eleven-dimensional supergravity.) This time,⁶³ the number of unbroken supersymmetries is determined by the number of *Killing spinors* on Y^{d-1} .

A more general ansatz which combines both of these ideas is a *warped product*: this is a background of the form $X^d \times_{\text{warped}} M^{1,9-d}$ with a metric

$$ds^2 = ds_X^2 + \phi(x)ds_M^2 \quad (42)$$

Table 9: Irreducible holonomy reps with covariantly constant spinor

Holonomy rep.	Geometry
$\{1\}$ on \mathbb{R}	flat
$SU(n)$ on \mathbb{R}^{2n} ($n \geq 3$)	Calabi–Yau
$Sp(n)$ on \mathbb{R}^{4n}	hyper-Kähler
G_2 on \mathbb{R}^7	G_2
$Spin(7)$ on \mathbb{R}^8	$Spin(7)$

with an appropriate (Ricci-flat) metric ds_X^2 , a flat metric ds_M^2 on M , and a conformal factor $\phi(x)$ depending on $x \in X$, accompanied by a non-trivial field strength for one of the supergravity fields. The space X should have finite volume, but might not be compact (due to the presence of branes).

To see why all of these constructions are related, note that anti-de Sitter space $AdS^{1,10-d}$ can be decomposed as a warped product of \mathbb{R}^+ and $M^{1,9-d}$. Then we can rewrite

$$Y^{d-1} \times AdS^{1,10-d} = (Y^{d-1} \times \mathbb{R}^+) \times_{\text{warped}} M^{1,9-d} \quad (43)$$

and the Killing spinors on Y^{d-1} go over to covariantly constant spinors on $Y^{d-1} \times \mathbb{R}^+$ (by a theorem of Bär⁶⁴). When Y^{d-1} is a sphere, we can regard this as a brane solution of the supergravity theory. For more general manifolds Y , there is an interpretation of this solution as corresponding to branes at singularities.⁶⁵

We will mainly focus on the case where X is compact and the spacetime is an ordinary product (not a warped product). In this case, the covariantly constant spinors are determined by the holonomy of the metric. (Similarly, the Killing spinors on Y in a Freund–Rubin ansatz will be determined by the “Weyl holonomy”—but the ordinary holonomy is easier to work with.)

The classification of holonomy groups of Riemannian manifolds is given by the Berger–Simons theorem^{66,67} (see the book of Besse⁶⁸

Table 10: Irreducible holonomy reps without covariantly constant spinor

Holonomy rep.	Geometry
$\mathrm{SO}(n)$ on \mathbb{R}^n	general Riemannian
$\mathrm{U}(n)$ on \mathbb{R}^{2n}	Kähler
$\mathrm{Sp}(1) \times \mathrm{Sp}(n)/\mathbb{Z}_2$ on \mathbb{R}^{4n}	quaternion Kähler
H on $\mathfrak{g}/\mathfrak{h}$	locally symmetric space G/H

for a complete account of holonomy). Actually, it is important to bear in mind that there is a holonomy *representation* which is being classified, not just a group. If we start at a point $x \in X$ and follow a loop which begins and ends at x , parallel transport along that path will transport tangent vectors at x along to tangent vectors at intermediate points, finally reaching a tangent vector at x again. This gives a mapping from $T_{X,x}$ to itself, and the group generated by all such mappings is the holonomy group (with $T_{X,x}$ giving the holonomy representation space). Note that parallel transport can also be applied to differential forms and to spinors (in the case of a spin manifold), so once the holonomy group is known, determining the covariantly constant differential forms or spinors is a simple exercise in representation theory. We give the *Berger–Simons classification of irreducible holonomy representations* in Tables 9 and 10. The holonomy representations listed in Table 9 are relevant for supersymmetric compactification of string theories: each has a covariantly constant spinor, and each is Ricci flat. The remaining holonomy representations (with no covariantly constant spinor) are listed in Table 10; these find an application in the study of moduli spaces of supersymmetric vacua. (This latter application is discussed in detail in Paul Aspinwall’s lectures in this volume.)

To apply this, we also need to know that every compact Riemannian manifold with a covariantly constant spinor admits a finite (unbranched) cover which can be decomposed as a Riemannian

product of a flat torus and a collection of compact Riemannian manifolds with irreducible holonomy representations. The first step in showing this is furnished by *de Rham's holonomy theorem*.⁶⁹ If a Riemannian manifold (M, g) is complete, simply connected and if its holonomy representation is reducible, then (M, g) is a Riemannian product. (It follows easily that if the original manifold had a covariantly constant spinor, then so does each factor in the de Rham decomposition, and as a consequence the metric on the manifold is Ricci flat.) The second step is the *Cheeger–Gromoll theorem*.⁷⁰ If (M, g) is a complete connected Riemannian manifold with non-negative Ricci curvature which admits a line, then (M, g) is a Riemannian product $(\overline{M} \times \mathbb{R}, \overline{g} \times dt^2)$ where $(\overline{M}, \overline{g})$ is a complete connected Riemannian manifold with non-negative Ricci curvature, and dt^2 is the canonical metric on \mathbb{R} . (Applying this result several times then yields the desired decomposition.⁷¹)

12 Supersymmetric string compactifications

If we are interested in compactifications of string theories (or M-theory) which preserve *some* supersymmetry, we should focus on the flat, Calabi–Yau, hyper-Kähler, G_2 , and $\text{Spin}(7)$ cases. The last two “exceptional” cases are poorly understood, and will not be discussed further here. (However, there has been some progress in understanding these manifolds—see the recent book of Joyce⁷² and references therein.) The flat case leads to the study of compact tori, which we have already described in lecture I.

The Calabi–Yau and hyper-Kähler manifolds can be given the following general characterizations (we assume the manifolds are *compact*):

Calabi–Yau manifolds (holonomy $\text{SU}(n)$, $n \geq 3$) have a non-vanishing holomorphic n -form Ω and a Kähler metric. There is a unique complex structure (up to complex conjugation) compatible with the metric. The Kähler metric can be described in terms of

the *Kähler form* $\omega = \frac{i}{2} \sum g_{z_i \bar{z}_j} dz_i \wedge d\bar{z}_j$. (Ω has a local description of the form $f(z) dz_1 \wedge \cdots \wedge dz_n$ with $f(z)$ holomorphic.)

Hyper-Kähler manifolds (holonomy $\mathrm{Sp}(n)$) have real dimension $4n$, with a distinguished three-plane of two-forms, and an S^2 of compatible complex structures. If we choose one of the complex structures, there is a holomorphic two-form of the form $\omega_1 + i\omega_2$ and a Kähler form $r\omega_3$ for some orthogonal basis $\omega_1, \omega_2, \omega_3$ of the three-plane, and some positive constant r . The manifold also has a holomorphic 4-form, 6-form, \dots , $2n$ -form given by taking powers of $\omega_1 + i\omega_2$. In particular, there is a form $\Omega = (\omega_1 + i\omega_2)^n$ of top degree. It is non-vanishing.

The metrics in all of these cases are Ricci-flat. Such metrics were studied by Calabi in the 1950's who showed that for a given complex structure and de Rham cohomology class of Kähler metrics, there is at most one Ricci-flat metric in the class. (That is, if ω is the Kähler form of a Ricci-flat metric, then there is no one-form η on X such that $\omega + d\eta$ is also the Kähler form of a Ricci-flat metric.)

Calabi conjectured⁷³ the existence of such metrics, and this was proved by Yau⁷⁴ in the 1970's in the following form: given a compact complex manifold X of complex dimension n which admits a non-vanishing holomorphic n -form Ω , and given a Kähler form ω on X , there exists a Ricci-flat metric on X whose Kähler form is in the same de Rham class as ω , and for which Ω is covariantly constant.

The proof is a non-constructive existence proof. In particular, although we are certain that these metrics exist, it is very difficult to calculate any of their properties.

However, this theorem is very powerful as a tool for studying string backgrounds, since it reduces the search for solutions to the supergravity equations of motion to a search for complex Kähler manifolds which have a non-vanishing holomorphic n -form Ω .

In fact, the search can be restricted even further:⁷¹ it turns

out that for every compact $SU(n)$ holonomy manifold ($n \geq 3$), the complex structure is *algebraic* (i.e., X comes from algebraic geometry); for hyper-Kähler manifolds, generically if you fix the Ricci-flat metric there will be choices out of the S^2 of complex structures for which X is algebraic.

So we can restrict our search to algebraic geometry, and employ a completely different set of tools to find and study such objects.

13 Algebraic geometry: a brief introduction

The “algebraic varieties” we now must study are complex submanifolds X of complex projective space \mathbb{P}^N . We describe \mathbb{P}^N by means of “homogeneous coordinates” $[z_1, z_2, \dots, z_N] \neq [0, 0, \dots, 0]$ which do not label points uniquely but are subject to identifications

$$[z_0, z_1, \dots, z_N] = [\lambda z_0, \lambda z_1, \dots, \lambda z_N] \quad (44)$$

for non-zero complex numbers λ . (We use square brackets to emphasize that these are not ordinary coordinates.)

Given $X \subseteq \mathbb{P}^N$, each homogeneous coordinate z_i determines a codimension one subvariety

$$D_i = X \cap \{z_i = 0\} \quad (45)$$

on X . (We are assuming that $X \not\subseteq \{z_i = 0\}$; otherwise, we would have treated X as a submanifold of $\{z_i = 0\} \cong \mathbb{P}^{N-1}$.) Such a codimension one subvariety is called an *effective divisor* on X . More generally, a combination $\sum m_i D_i$ of effective divisors with integer coefficients is called a *divisor*.

If we consider two of these divisors, D_i and D_j , the ratio z_i/z_j makes sense as a function on $X - D_i - D_j$. (The individual homogeneous coordinates are not functions on X or on \mathbb{P}^N due to the identifications in Eq. ((44)), but the identifications cancel out in ratios.) This ratio z_i/z_j extends to a *meromorphic function* on X : its only singularities are poles.

Generally, for a meromorphic function f defined on X , we define the *divisor of f* to be

$$\operatorname{div}(f) = \{f = 0\} - \{f = \infty\} \quad (46)$$

where $\{f = 0\}$ and $\{f = \infty\}$ are codimension one in X . In the example at hand, we have

$$\operatorname{div}(z_i/z_j) = D_i - D_j. \quad (47)$$

This property is characteristic of divisors which occur as intersections with linear functions in \mathbb{P}^N for the *same* embedding in projective space. In general, a given algebraic variety will have many different embeddings into projective spaces.

To determine all of the ways to embed X into projective spaces, we can study *all* of the divisors on X . To determine which divisors belong to the same embedding, we introduce some definitions. Two divisors D and D' are said to be *linearly equivalent* if there is a meromorphic function f such that

$$\operatorname{div}(f) = D - D'. \quad (48)$$

The *linear system containing D* is the set

$$\begin{aligned} |D| = \{D' \mid D' \text{ is linearly equivalent to } D, \text{ and } D' = \sum n_i D_i \\ \text{with } n_i \geq 0 \text{ and } D_i \subseteq X \text{ codimension one}\}. \end{aligned} \quad (49)$$

The last requirement in the definition come from the observation that the divisors we encountered from $X \subseteq \mathbb{P}^N$ were effective divisors, i.e., subsets of X counted with multiplicity, but with no negative coefficients allowed.

Given a linear system $|D|$, we choose a basis $D_0 = D, D_1, \dots, D_n$ of the divisors in $|D|$, and let f_1, f_2, \dots, f_n be the meromorphic functions satisfying

$$\operatorname{div}(f_j) = D_j - D_0. \quad (50)$$

Then we can define a mapping $X \rightarrow \mathbb{P}^n$ by

$$x \mapsto [1, f_1(x), f_2(x), \dots, f_n(x)]. \quad (51)$$

This is ill-defined along D_0 , but by exploiting the equivalence in \mathbb{P}^n we can rewrite this as

$$[1, f_1(x), f_2(x), \dots, f_n(x)] = [\frac{1}{f_1(x)}, 1, \frac{f_2(x)}{f_1(x)}, \dots, \frac{f_n(x)}{f_1(x)}] \quad (52)$$

which is ill-defined along D_1 instead of along D_0 , and so on, for other divisors D_j .

Thus, if the divisors D_0, D_1, \dots, D_n have no points in common, our prescription Eq. (51) can be extended to a well-defined mapping on all of X . In this case, $|D|$ is said to be *base point free*.

The linear system $|D|$ is said to be *very ample*^g if the associated mapping is actually an embedding into \mathbb{P}^n . Given a very ample linear system $|D|$, i.e., an embedding $X \subseteq \mathbb{P}^n$, we get a natural Kähler metric on X by restricting the Fubini–Study metric from \mathbb{P}^n . Explicitly, the Kähler form of this metric on \mathbb{P}^n can be written $\omega = \partial\bar{\partial} \log \sum |z_i|^2$. Restricting to X , we get a form $\omega_{|D|} = (\partial\bar{\partial} \log \sum |z_i|^2)|_X$ on X .

A key fact is that for Calabi–Yau manifolds, the Kähler classes $\omega_{|D|}$ coming from projective embeddings will generate all Kähler classes (using positive real linear combinations).^h So this portion of our problem—determining the set of Kähler classes—can be solved using algebraic geometry. (The hyper-Kähler case is different and will be discussed in lecture IV.)

The other portion of our problem—determining the set of complex structures—is also a problem in algebraic geometry. Once X has been embedded in \mathbb{P}^n , it can always be described by means of a finite set of homogeneous equations

$$f_1(z_0, \dots, z_n), \dots, f_k(z_0, \dots, z_n), \quad (53)$$

^gA linear system $|D|$ is *ample* if some positive multiple $|mD|$ is very ample.

^hThis is because $h^{2,0} = 0$, so the positive rational linear combinations of very ample classes will be dense in the Kähler cone.

with

$$X = \{[z_0, \dots, z_n] \in \mathbb{P}^n \mid f_j(z_0, \dots, z_n) = 0 \text{ for all } j\}. \quad (54)$$

In principle, the other complex structures are found by varying the coefficients in these defining equations. There are two difficulties with this in practice:

1. There may be some complex structures on this manifold with don't embed into the same projective space as X .
2. The number of equations needed to describe X is larger than $\dim \mathbb{P}^n - \dim X$, and the equations don't meet transversally; thus, if we vary the coefficients arbitrarily we will find a common intersection which is smaller than X . (So we must vary the coefficients *judiciously*, and it is hard to see explicitly how to do this.)

We will encounter both of these phenomena in our discussion of K3 surfaces below. Over the years, algebraic geometers have developed some rather sophisticated machinery to address these issues. (See, for example, the treatise of Viehweg.⁷⁵) Very little of this machinery has been applied to cases of interest in physics (to date!).

So we have seen that the complex structures can be studied by varying coefficients, and the Kähler classes can be studied by locating all (very ample) divisors. The issue we have not yet addressed is: how can we recognize whether or not there exists a non-vanishing holomorphic n -form?

A very useful tool in studying this issue is the “adjunction formula.” Given a complex submanifold $D \subseteq X$ defined by a single equation $f = 0$ (locally), there is a “Poincaré residue formula” relating meromorphic n -forms on X and meromorphic $(n - 1)$ -forms on D : given a meromorphic n -form

$$\frac{g(w_1, \dots, w_n) dw_1 \wedge \dots \wedge dw_n}{f(w_1, \dots, w_n)} \quad (55)$$

with a simple pole on D (using local coordinates w_1, \dots, w_n on X), its Poincaré residue is

$$\left. \frac{g(w)dw_1 \wedge \dots \wedge dw_{n-1}}{\partial f / \partial w_n} \right|_D \quad (56)$$

(which is well-defined if $\partial f / \partial w_n \neq 0$) with similar, equivalent, formulas when $\partial f / \partial w_j \neq 0$. (If D is a submanifold, then at every point one of the $\partial f / \partial w_j$'s must be $\neq 0$.)

It is common to express the properties of meromorphic n -forms in terms of *divisors*; if $\alpha(w_1, \dots, w_n)dw_1 \wedge \dots \wedge dw_n$ is a meromorphic n -form, we define the *canonical divisor of X* to be

$$K_X = \text{div}(\alpha) = \{\alpha = 0\} - \{\alpha = \infty\}. \quad (57)$$

Thus, in our Poincaré residue formula, we see

$$K_X = \text{div}(g) - \text{div}(f) = \text{div}(g) - D \quad (58)$$

while

$$K_D = \text{div}(g)|_D \quad (59)$$

(since $\partial f / \partial w_j \neq 0$). Thus,

$$K_D = (K_X + D)|_D. \quad (60)$$

This is known as the *adjunction formula*.

The interpretation of $D|_D$ is this: find a divisor D' which is linearly equivalent to D , and treat $D'|_D$ as a divisor on D . (All facts about these divisors are being considered up to linear equivalence only.) This is the divisorial version of the “normal bundle” of D .

The requirement in Yau’s theorem is that there exists a meromorphic n -form whose divisor is trivial, i.e., it has neither zeros nor poles; this is the same as saying that $K_X = 0$.

Key example. $K_X = 0$ and $D \subseteq X$ is a codimension one submanifold. The adjunction formula tells us that $K_D = D|_D$.

Note that $D|_D$ becomes quite concrete if we embed X in \mathbb{P}^n using the linear system $|D|$: then $D'|_D$ represents the intersection of D with some $z_i = 0$. In other words, D is embedded by the canonical linear system $|K_D|$.

14 Algebraic geometry of K3 surfaces

The case of $\dim_{\mathbb{C}} X = 2$ (an algebraic K3 surface) is instructive. In this case, D is a Riemann surface, which must have some genus g . The degree of the canonical divisor of a Riemann surface is well-known: $\deg(K_D) = 2g - 2$. Also, the canonical linear system $|K_D|$ embeds D into \mathbb{P}^{g-1} .

The interpretation of these facts in terms of X is that X should embed in \mathbb{P}^g , and its *degree* (the number of points of intersection $X \cap \{z_i = 0\} \cap \{z_j = 0\}$) should be $2g - 2$.

Remarkably, surfaces $X \subseteq \mathbb{P}^g$ of this type exist for every $g \geq 2$, and in every case, the number of independent deformations of complex structure is 19. These are the *algebraic K3 surfaces*.

Let us consider these surfaces for low values of g .

$g = 2$ Riemann surfaces of genus two are hyperelliptic, and map two-to-one onto \mathbb{P}^1 . So X will map two-to-one onto \mathbb{P}^2 . The map on D must have six branch points (in order to get genus two), so the map $X \rightarrow \mathbb{P}^2$ must be branched over a curve of degree six. We can describe X by an equation of the form

$$y^2 = z_0^6 + z_1^6 + z_2^6 + \cdots \quad (61)$$

(the degree six equation on the right hand side can be arbitrary), and regard this as an equation in a *weighted projective space* $\mathbb{P}^{1,1,1,3}$ in which $[z_0, z_1, z_2, y] = [\lambda z_0, \lambda z_1, \lambda z_2, \lambda^3 y]$. (The superscripts in the notation denote the powers of λ , the so-called *weights* of the homogeneous variables.)

$g = 3$ The general Riemann surface of genus three embeds as a degree four curve in \mathbb{P}^2 ; X should be a surface of degree four in \mathbb{P}^3 , for example,

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0. \quad (62)$$

$g = 4$ This time, $D \subseteq \mathbb{P}^3$ is the intersection of surfaces of degrees two and three, and $X \subset \mathbb{P}^4$ will be the intersection of hypersurfaces of degrees two and three. The degree is then $2 \cdot 3 = 6$.

$g = 5$ $D \subseteq \mathbb{P}^4$ and $X \subseteq \mathbb{P}^5$ can be described as the intersection of three hypersurfaces of degree two. The degree is $2 \cdot 2 \cdot 2 = 8$.

$g \geq 6$ $D \subseteq \mathbb{P}^{g-1}$ and $X \subseteq \mathbb{P}^g$ require more defining equations than their codimension. This makes the moduli problem tricky—as indicated above, coefficients must be varied *judiciously*.

One interesting feature to note about this set of examples: the complex dimension of the space of Riemann surfaces of genus g is $3g - 3$, whereas the dimension of those which lie on a K3 surface is at most $19 + g$ (19 parameters for X and g parameters for the choice of D when $X \subseteq \mathbb{P}^g$). Thus, when $g > 11$, not every curve lies on a K3 surface.

Another interesting feature, to be discussed further in lecture IV, is that the set of all complex structures on a K3 surface has complex dimension 20, and form a single family containing all of the algebraic K3 surfaces of every g . This is an example of the phenomenon mentioned above in which not all deformations of complex structure may happen in the given projective space.

However, when the holonomy is $SU(n)$, $n \geq 3$, there always exist embeddings $X \subseteq \mathbb{P}^N$ for which all nearby complex structures can be obtained within the same \mathbb{P}^N . (Warning: if somebody hands you $X \subseteq \mathbb{P}^N$, it might not have this property: some projective embeddings are “deficient” in this sense.)

15 Calabi–Yau manifolds in higher dimension

The theory of K3 surfaces is understood in great detail, and will be explained further in the next lecture. We know much less about the algebraic geometry of Calabi–Yau or hyper-Kähler manifolds of higher dimension. There are two strategies which might be followed:

1. try to directly generalize constructions like the $g \leq 5$ cases of K3 surfaces
2. try to study in general the possible divisors D and whether they occur on Calabi–Yau or hyper-Kähler manifolds.

The first strategy has led to an extensive study of Calabi–Yau “complete intersections” in projective spaces, and more generally in weighted projective spaces or toric varieties (a further generalization of weighted projective space). At least tens of thousands of examples have been produced in this way.ⁱ

And yet, as the above story about K3 surfaces illustrates, such constructions may have only barely scratched the surface.

It is instructive to see why the set of complete intersection Calabi–Yau manifolds (of fixed dimension) in projective space is finite. Suppose $X^d \subseteq \mathbb{P}^n$ has been defined as the intersection

$$X = Y_1 \cap Y_2 \cap \cdots \cap Y_{n-d} \tag{63}$$

of $n - d$ hypersurfaces. Each Y_j is linearly equivalent to $m_j H$, where $H = \{z_0 = 0\}$, and m_j is the degree of the homogeneous polynomial defining Y_j . We use that fact that $K_{\mathbb{P}^n} = -(n + 1)H$ (which can be seen from the existence of a globally well-defined

ⁱFor example, there are 473,800,776 types of Calabi–Yau hypersurfaces in four-dimensional toric varieties, which give at least 30,108 distinct examples, based on Hodge numbers.⁷⁶

meromorphic n -form

$$\frac{z_0 dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n + z_1 dz_2 \wedge \cdots \wedge dz_n \wedge dz_0 + \cdots}{z_0 z_2 \cdots z_{n-1} z_n} \quad (64)$$

with poles along all $n + 1$ coordinate hyperplanes) and apply the adjunction formula repeatedly:

$$\begin{aligned} K_X &= (\cdots ((K_{\mathbb{P}^n} + Y_1)|_{Y_1} + Y_2)|_{Y_2} + \cdots + Y_{n-d})|_{Y_{n-d}} \\ &= (-(n+1)H + m_1 H + m_2 H + \cdots + m_{n-d} H)|_X. \end{aligned} \quad (65)$$

So the condition to get a Calabi–Yau or hyper-Kähler manifold is: $\sum_{j=1}^{n-d} m_j = n + 1$. Moreover we can choose $m_j \geq 2$ for each j since otherwise X would sit in a linear subspace (a smaller \mathbb{P}^n). So the condition can be rewritten as

$$\sum_{j=1}^{n-d} (m_j - 1) = d + 1, \quad m_j - 1 \geq 1 \quad (66)$$

and with a fixed d there are clearly only a finite number of solutions. The K3 examples described above are reproduced by the solutions $3 = 3$, $3 = 2 + 1$, $3 = 1 + 1 + 1$, giving degrees 4, $(3, 2)$, and $(2, 2, 2)$, respectively, with hyperplane sections having $g = 3$, $g = 4$, and $g = 5$.

The second strategy would seem to be a more general one: first, we study all surfaces D for which $|K_D|$ gives an embedding (or at least a reasonable map, like the two-to-one map we encountered for Riemann surfaces of genus two), and then we try to decide which ones can be on Calabi–Yau threefolds. As the remark about not all curves lying on K3 surfaces indicated, the second part will be highly non-trivial. However, even the first part is quite hard.

For curves, we had a simple invariant (the genus), and rather complete knowledge about the set of curves of genus g . When D is a surface, there are several invariants, including the Euler

number $e(D) = \chi_{\text{top}}(D)$ and the degree of the canonical divisor $c_1^2 = \#(K_D \cap K_D)$. It is convenient to introduce

$$\chi(\mathcal{O}_D) = \frac{e(D) + c_1^2}{12}, \quad (67)$$

which is an integer, and to use $\chi(\mathcal{O}_D)$ in place of $e(D)$. There are inequalities which constrain these invariants to the region bounded by $c_1^2 = 9\chi(\mathcal{O}_D)$, $c_1^2 = 0$, and $c_1^2 = 2\chi(\mathcal{O}_D) - 6$, as illustrated in Figure 5. Surfaces are generally less numerous above the central line $c_1^2 = 8\chi(\mathcal{O}_D)$ in the Figure than below it.⁷⁷

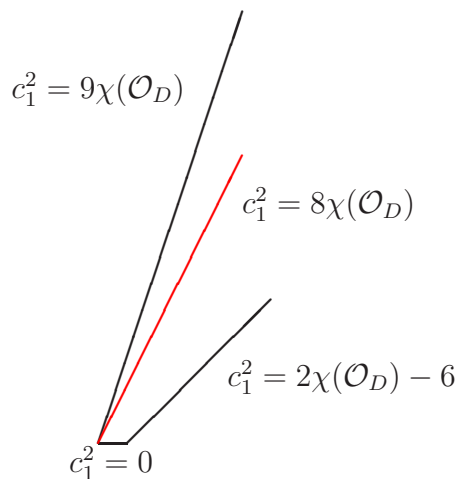


Figure 5: Constraints on invariants of surfaces of general type

For each point on the graph, there are at most a finite number of families, but it is not known how many, nor what are their dimensions, etc. See Persson⁷⁸ for a survey of what *is* known.

It has often been speculated that the number of families of Calabi–Yau threefolds might be finite. Certainly, the vast array of possibilities of D , together with the phenomenon of algebraic K3 surfaces for every $g \geq 2$ casts some doubt on this. (However,

as we shall see, the K3 surfaces are in fact unified into a single family of hyper-Kähler manifolds.) Of course, many Calabi–Yau threefolds have a wide variety of divisors D on them, so there will be much duplication. At the moment, it’s hard to tell whether the expectation that the number of families is finite is reasonable or not.

Lecture IV: K3 Duality

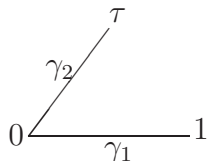
16 Flat metrics on a two-torus

In the previous lecture, we did not discuss the case of a torus T^d in any detail. As we had earlier seen (lecture I), the set of flat metrics on T^d admits a simple description:

$$\text{Met}(T^d) = \text{SL}(d, \mathbb{Z}) \backslash \mathbb{R}^\times \times \text{SL}(d, \mathbb{R}) / \text{SO}(d) \quad (68)$$

so the techniques of algebraic geometry were not needed. However, it is useful to see how algebraic geometry can be used to analyze this in the case $d = 2$.

The conformal class of a metric on T^2 is equivalent to the choice of complex structure. Traditionally, one describes the complex structure by representing T^2 as $\mathbb{C}/\langle 1, \tau \rangle$, i.e., $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$, where the choice of $\tau \in \mathfrak{h}$ (the upper half plane) specifies the periodicity. The one-cycles γ_1 and γ_2 which are represented by curves in \mathbb{C} joining the origin to 1 and to τ , respectively,



give a basis of the first homology. The torus has a holomorphic one-form, represented by dz in these coordinates, and the integrals over the generating cycles γ_1, γ_2 give $\int_{\gamma_1} dz = 1, \int_{\gamma_2} dz = \tau$. When we change basis of $H_1(T^2, \mathbb{Z})$ using $\text{SL}(2, \mathbb{Z})$, we get the standard

$\mathrm{SL}(2, \mathbb{Z})$ action on the upper half plane. (Note that dz is only unique up to a constant multiple:

$$\int_{\gamma_1} A \cdot dz = A, \quad \int_{\gamma_2} A \cdot dz = A\tau \quad (69)$$

so the truly invariant quantity is the ratio $\int_{\gamma_2} dz / \int_{\gamma_1} dz = \tau$.)

Since $\mathfrak{h} = \mathrm{SL}(2, \mathbb{R}) / \mathrm{SO}(2)$, we recover the description of the moduli space $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathfrak{h}$.

To relate this to algebraic geometry, we need to study meromorphic functions on $\mathbb{C} / \langle 1, \tau \rangle$. The basic such function was studied by Weierstrass in the 19th century, called the Weierstrass \mathfrak{p} -function:

$$\mathfrak{p}(z) = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left(\frac{1}{(z + m + n\tau)^2} - \frac{1}{(m + n\tau)^2} \right). \quad (70)$$

This is doubly-periodic, with periods 1, τ , and has a double pole at every lattice point, so descends to a meromorphic function on $\mathbb{C} / \langle 1, \tau \rangle$ with a double pole at the origin.

Weierstrass found a remarkable relation which this function satisfies:

$$(\mathfrak{p}'(z))^2 = 4(\mathfrak{p}(z))^3 - 60G_4\mathfrak{p}(z) - 140G_6, \quad (71)$$

where

$$G_4 = \sum_{(m,n) \neq (0,0)} \frac{1}{(m + n\tau)^4}, \quad G_6 = \sum_{(m,n) \neq (0,0)} \frac{1}{(m + n\tau)^6}. \quad (72)$$

If we define $y = \mathfrak{p}'(z)$, $x = \mathfrak{p}(z)$ and regard $[1, x, y]$ as in \mathbb{P}^2 , we find a cubic curve

$$y^2 = 4x^3 - 60G_4x - 140G_6 \quad (73)$$

which is identical with $\mathbb{C} / \langle 1, \tau \rangle$. In homogeneous coordinates, this becomes

$$y^2w = 4x^3 - 60G_4xw^2 - 140G_6w^3 \quad (74)$$

and the point $[0, 0, 1]$ was added representing the double pole of $\mathfrak{p}(z)$.

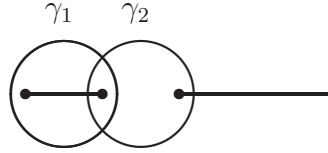
This process can be reversed: given a cubic curve E

$$y^2 = 4x^3 - ax - b \quad (75)$$

there is a non-vanishing holomorphic one-form given by the residue of

$$\frac{2dx \wedge dy}{y^2 - 4x^3 + ax + b} \quad (76)$$

(i.e., by the adjunction formula, $K_E = (K_{\mathbb{P}^2} + 3H)|_E = -3H + 3H|_E = 0$). The integration cycles are related to the branch cuts for the function $y = \sqrt{4x^3 - ax - b}$



and we find periods given by *elliptic integrals*

$$\int_{\gamma_j} \frac{dx}{y} = \int_{\gamma_j} \frac{dx}{\sqrt{4x^3 - ax - b}} \quad (77)$$

so that we recover

$$\tau = \int_{\gamma_2} \frac{dx}{\sqrt{4x^3 - ax - b}} \bigg/ \int_{\gamma_1} \frac{dx}{\sqrt{4x^3 - ax - b}}. \quad (78)$$

This is well-defined only up to $\text{SL}(2, \mathbb{Z})$.

17 Kodaira's classification, and F-theory/heterotic duality

We can now explain how Kodaira⁵² found the classification of singular fibers in one-parameter elliptic fibrations, mentioned in lecture II. If we consider families

$$y^2 = x^3 + f(t)x + g(t) \quad (79)$$

where now $f(t)$ and $g(t)$ are polynomials, we can try to classify all possible behaviors near $t = 0$ and the corresponding monodromy on the periods. Singularities occur when the cubic polynomial has multiple roots, and that is measured by the *discriminant*:

$$\Delta(t) = 4f(t)^3 + 27g(t)^2. \quad (80)$$

Kodaira's analysis classifies possible monodromies in terms of the divisibility properties $t^a \mid f(t)$, $t^b \mid g(t)$, $t^c \mid \Delta(t)$. It is a *local* analysis in t . For example, if t does not divide both $f(t)$ and $g(t)$, and $t^N \mid \Delta(t)$, then we are in Kodaira's case I_N (notation as in Table 7). The monodromy can be calculated by analyzing how the elliptic integrals in Eq. (77) depend on parameters.

The order of zero of $\Delta(t)$ measures the deficit angle in the corresponding stringy cosmic string metric, with an angle of $\pi m/6$ when $t^m \mid \Delta(t)$, $t^{m+1} \nmid \Delta(t)$.

Thus, to find a global solution to the stringy cosmic string metric, with total deficit angle 4π , we need a total of 24 zeros of $\Delta(t)$. This comes about precisely when $\deg f(t) = 8$, $\deg g(t) = 12$.

Let us rewrite Eq. (79) in homogeneous form, as a surface S in a \mathbb{P}^2 bundle over \mathbb{P}^1 , with coordinates $[w, x, y]$ on \mathbb{P}^2 and $[s, t]$ on \mathbb{P}^1 . It takes the form

$$y^2w = x^3 + f_8(s, t)xw^2 + g_{12}(s, t)w^3. \quad (81)$$

Adapting the adjunction formula to this situation shows that $K_S = 0$, i.e., a stringy cosmic string solution which can be used to build an F-theory model corresponds to a K3 surface when the τ -function is realized by elliptic curves. Thus, we expect that when F-theory models in eight dimensions are compactified on a circle, the result is M-theory on a K3 surface. (We have already encountered this possibility in our remarks about T-dualizing type I on T^3 .) This leads to the first of the K3 duality statements: *F-theory from K3 is dual to the heterotic string on T^2 .*

Going up one dimension, we can ask if there is a K3 interpretation for the nine-dimensional heterotic string models. Such an interpretation has been proposed⁵⁰ as a limit in which the S^2 stretches to a long cylinder. A more precise realization of this picture has recently been worked out by Cachazo and Vafa⁴⁴—it involves *real* K3 surfaces, i.e., restricting the coefficients in Eq. (79) to real numbers and searching for real solutions.

18 M-theory/heterotic duality

To find K3 duality statements in lower dimension, we need to study the Ricci-flat metrics on K3, or more generally on hyper-Kähler manifolds.⁷¹

Let X^{4d} be a hyper-Kähler manifold. A somewhat abstract description of the Ricci-flat metrics on X goes like this:⁷⁹ $H^2(X, \mathbb{R})$ has a natural inner product

$$\begin{aligned} H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R}) &\rightarrow \mathbb{R} \\ (\alpha, \beta) &\mapsto \int_X \alpha \wedge \beta \wedge \omega^{2d-2} \end{aligned} \quad (82)$$

(which only changes by a scale factor if the Kähler form ω is changed). The *signature* of this inner product is $(3, k)$, where $b_2(X) = 3 + k$ depends on X . (For a K3 surface, $k = 19$.)

Each Ricci-flat metric on X determines a positive three-plane in $H^2(X, \mathbb{R})$, the “self-dual” harmonic two-forms (with respect to the inner product above). The space of all positive three-planes is

$$\Gamma \backslash \mathrm{O}(3, k) / \mathrm{O}(3) \times \mathrm{O}(k), \quad (83)$$

where we have taken the quotient by Γ which comes from the diffeomorphism group of X . So there is a map

$$\{\text{Ricci-flat metrics}\} \rightarrow (\Gamma \backslash \mathrm{O}(3, k) / \mathrm{O}(3) \times \mathrm{O}(k)) \times \mathbb{R}^+ \quad (84)$$

(with the last \mathbb{R}^+ representing the volume). However, it is very difficult to directly determine whether this map is one-to-one or onto. It *is* possible to show that locally, the map is one-to-one and onto, i.e., small variations of the Ricci-flat metric are accurately reflected by small variations of the positive three-plane and the volume.

To give a more concrete interpretation of this space, and relate it to Yau's theorem and algebraic geometry, we must choose a complex structure compatible with the Ricci-flat metric. Rather than study the totality of all such complex structures, we will select an element $\lambda \in H^2(X, \mathbb{Z})$ for which $\int_X \lambda \wedge \lambda \wedge \omega^{2d-2} > 0$, and choose the complex structure so that λ becomes type $(1, 1)$. To do that, if Π denotes our three-plane, then $\lambda^\perp \cap \Pi$ is a two-plane; we let ω_1, ω_2 be a basis of that two-plane and use the complex structure for which $\omega_1 + i\omega_2$ is holomorphic (and $\omega_3 \in \omega_1^\perp \cap \omega_2^\perp \cap \Pi$ is a Kähler form).

The advantage of this choice is that λ will now be an algebraic class, since it is integral and type $(1, 1)$, and for general moduli some multiple $m\lambda$ will be a *very ample class*. Thus, we can study hyper-Kähler manifolds which are embedded in \mathbb{P}^n , with some specific type of embedding, and capture all the information we need about metrics on the space in that context.^j

With respect to our chosen complex structure (which we could describe algebraically, as in the case of all the algebraic K3 surfaces in lecture III), the set of all Kähler classes will correspond to the set of Ricci-flat metrics, by Calabi's and Yau's theorems. However, unlike the Calabi–Yau case we cannot give a purely algebraic description of the Kähler classes. The defining conditions we need, for possible Kähler classes κ (with respect to our complex structure)

^jNote that we are getting complete information only about the open subset in the moduli space where $m\lambda$ is very ample; if we vary λ and/or m , we can change this open set. The full moduli space will be the union of open sets of this kind.

are:

1. $\kappa \cdot \omega_1 = 0$
2. $\kappa \cdot \omega_2 = 0$
3. $\kappa \cdot \kappa > 0$
4. $\int \kappa^\ell \alpha > 0$ for every α representing the cohomology class of a complex submanifold of X of complex codimension ℓ .

Conditions 1 and 2 restrict κ to a space of dimension $b_2 - 2 = k + 1$; conditions 3 and 4 select an open subset and do not affect the dimension.

Moreover, it is known that the set of algebraic deformations of X has complex dimension k (and the set of *all* complex deformations has complex dimension $k + 1$). As a check, then, counting real parameters for our metric we find $2k$ from the algebraic deformations of complex structure and $k + 1$ from the compatible Kähler metrics, for a total of

$$3k + 1 = \dim_{\mathbb{R}} (\mathbb{R}^+ \times \mathrm{O}(3, k) / (\mathrm{O}(3) \times \mathrm{O}(k))) . \quad (85)$$

Focusing on algebraic deformations allows us to apply some of the machinery from algebraic geometry to study the variations of complex structure. In the case of K3 surfaces, there is a “Torelli theorem” for this moduli space^k which tells us the precise answer: the set of metrics is given by

$$\mathrm{O}(3, 19; \mathbb{Z}) \setminus (\mathrm{O}(3, 19) / \mathrm{O}(3) \times \mathrm{O}(19) - Z) \times \mathbb{R}^+, \quad (86)$$

^kThe Torelli theorem for K3 surfaces has a long history. An account of the original theorem can be found in the “Séminaire Palaiseau.”⁸ The version stated here was proved directly by Anderson,⁸⁰ and can also be extended to the singular set Z .^{81, 82}

where

$$Z = \{\Pi \text{ such that } \Pi \cdot e = 0 \text{ for some } e \in H^2(X, \mathbb{Z}) \text{ with } e \cdot e = -2\}. \quad (87)$$

(Such planes $\Pi \in Z$ *cannot* correspond to metric, since $\pm e$ represents the class of a rational curve on X , which would have vanishing area in such a “metric.”)

As we saw when discussing the T-duals of type I on T^3 , we are expecting a duality between M-theory on K3 and the heterotic string on T^3 . For M-theory compactification, the scalars in the effective theory will be given by the set of Ricci-flat metrics on K3 as well as by harmonic three-forms. But K3 manifolds have no harmonic three-forms, so the entire M=theory moduli space is

$$O(3, 19; \mathbb{Z}) \setminus (O(3, 19) / O(3) \times O(19) - Z) \times \mathbb{R}^+. \quad (88)$$

This *agrees* with the heterotic string on T^3 , except for the phenomenon of the subset Z .

What is the interpretation of Z ? As described above, the problem we encounter along Z is that a holomorphic S^2 is shrinking to zero area. However, M-theory contains more than supergravity, and in particular we can wrap the M-theory membrane around this S^2 . The mass of the corresponding state is proportional to the area, so along Z we will find new massless states, corresponding to the e ’s with $e \cdot \Pi = 0$.

On the heterotic side, Z is the locus along which the gauge symmetry becomes non-abelian. Now we are finding the source of non-abelian gauge symmetry in M-theory: massless multiplets from shrinking S^2 ’s, which are *vectors* (new gauge fields) in the spectrum of the effective theory.

The geometry of configurations of e ’s is quite pretty. We need to study collections of holomorphic S^2 ’s (i.e., \mathbb{P}^1 ’s) embedded in X whose intersection matrix $(e_i \cdot e_j)$ is negative definite. The restriction to negative definite matrices arises because these e ’s lie

in Π^\perp , which is a negative definite space of dimension 19. The entries in the intersection matrix are $e_i \cdot e_i = -2$ (from the adjunction formula again, since $g = 0$), and $e_i \cdot e_j = 0$ or 1 when $i \neq j$. ($e_i \cdot e_j \geq 2 \implies (e_i + e_j)^2 \geq 0$, a contradiction.)

This is precisely the same combinatorial problem as the one which classified simply-laced Dynkin diagrams, and the answer is the same, illustrated in Figure 6. In the Figure, we have drawn the holomorphic curves—each of self-intersection -2 —and indicated which ones meet.

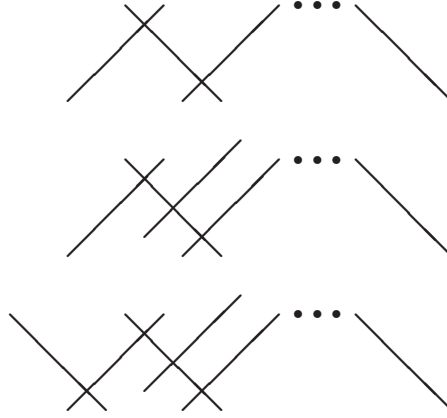


Figure 6: Dynkin diagrams for ADE groups

In M-theory, we will associate gauge algebras $\mathfrak{su}(n)$, $\mathfrak{so}(2n)$, \mathfrak{e}_6 , \mathfrak{e}_7 , \mathfrak{e}_8 to these cases. In fact, the set of classes $e = \sum m_i e_i$ for which $e \cdot e = -2$ precisely correspond to the roots in the Lie algebra, and we will get a gauge boson from the M-theory membrane wrapped around each such class.

The picture in algebraic geometry of these singular spaces is fairly benign; the coefficients of the equations have been tuned to special values at which such singularities appear. The singularities can be removed either by varying the complex structure, or by “blowing up” the algebraic variety which effectively increases the

area assigned to e_j .

One description of these singularities is that they are precisely the orbifolds \mathbb{C}^2/Γ where $\Gamma \subseteq \text{SU}(2)$ is a finite subgroup acting without fixed points away from the origin. The dictionary was already given in Table 8. Note that one lesson from this analysis is that M-theory on an orbifold is only well-behaved when the non-abelian gauge theory coming from the new massless vectors is included.

19 Type IIA/heterotic duality

When we extend this analysis to type IIA on K3, we find that the orbifold points will be accompanied by a choice of B -field value. When the B -field is zero, the perturbative string is singular, and the new massless states from wrapped D2-branes must be included. However, the theory is perturbatively non-singular with B -fields turned on; in particular, Aspinwall⁸³ checked that the B -field is non-zero at the orbifold conformal field theories, so string theory is indeed nonsingular (even at the perturbative level) at such points.

The summary of our K3 dualities is:

$$\text{het on } T^2 \leftrightarrow \text{F-theory from K3}$$

$$\text{het on } T^3 \leftrightarrow \text{M-theory on K3}$$

and, as we have just briefly indicated, this extends to

$$\text{het on } T^4 \leftrightarrow \text{IIA on K3}$$

in a similar manner.

The final duality is bolstered by a construction of a soliton of the type IIA string theory compactified on K3,⁸⁴ which in the appropriate limit exhibits the characteristics of a heterotic string.

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